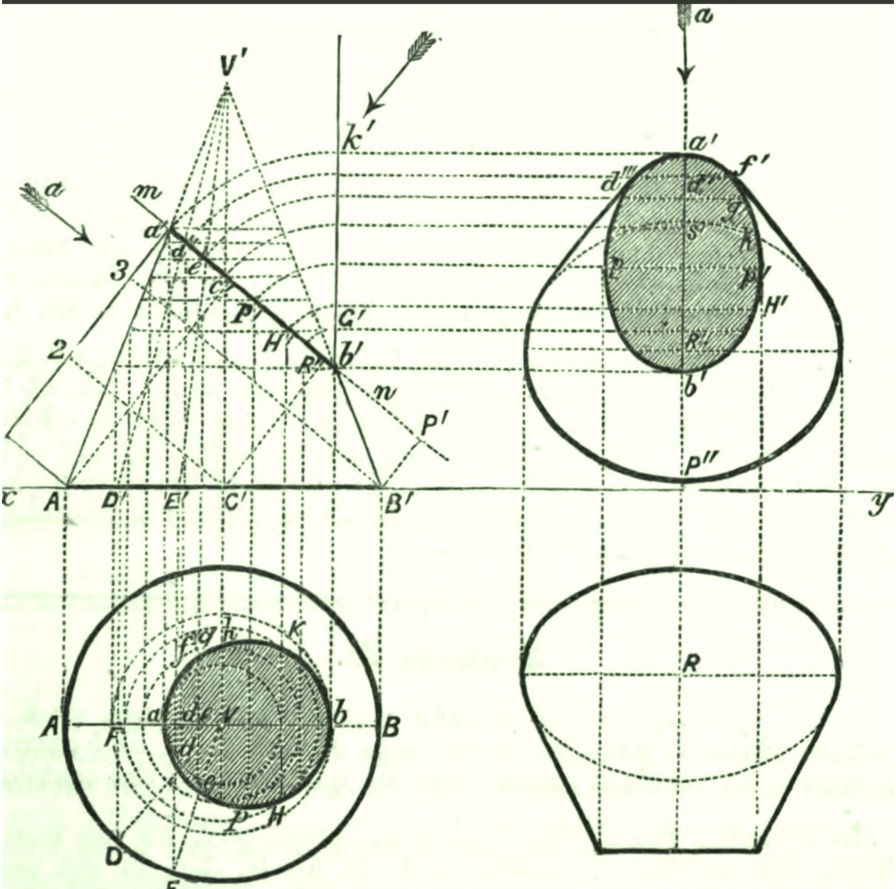


A graduated course of problems in practical plane and solid ...

James Martin

A GRADUATED COURSE

OF

PROBLEMS IN PRACTICAL PLANE AND SOLID GEOMETRY,

TOGETHER WITH

MISCELLANEOUS EXERCISES IN PRACTICAL PLANE AND SOLID GEOMETRY;
ETYMOLOGY OF GEOMETRICAL TERMS, &c. &c.

BY

JAMES MARTIN,

HEAD MASTER OF THE ENDOWED SCHOOL, WEDGWOOD INSTITUTE, BURSLEM.

Author of "Elements of Euclid," &c.

LONDON:
GEORGE PHILIP & SON, 32 FLEET STREET;
LIVERPOOL: CAXTON BUILDINGS, SOUTH JOHN STREET,
AND 49 & 51 SOUTH CASTLE STREET.
1876.

PREFACE.

THE present volume on Practical Geometry will be found to be suitable both for the Art-student and the Art-workman.

It falls into two distinct parts, viz., *Plane* and *Solid.*

It will be seen that though the problems in the *Plane* portion of the work are exceedingly numerous, they are classified in sections.

The *definitions* which precede the several sections, should be thoroughly mastered by the pupil before entering on the problems.

The *diagrams* are engraved with extreme care, and as is usual in works on Practical Geometry, *three* kinds of lines are used, viz., (1) *thin* lines, representing those which are given; (2) *dotted* lines, representing those used in the construction of the figure; and (3) *thick* lines, representing the solution of the problem.

Moreover, the two cardinal ideas, viz., what is *given,* and what is *to be done*, are by a typographical expedient shown also in the enunciation.

Respecting the *solid* portion of the work, it will be found that the problems are carefully graduated, and are also arranged in sections.

In conformity with the usual practice, the *base-line* is always represented by the letters *xy*, and the *style* of lettering uniformly indicates whether a point, line, &c., is in space, in the vertical plane, or in the horizontal.

In order to make the geometrical terms used more instructive, a section on their derivation has been incorporated in the work.

Finally, in order to make the volume complete, it closes with an index of all the problems given in both parts.

JAMES MARTIN.

WEDGWOOD INSTITUTE,
BURSLEM.

CONTENTS.

PAGE

On Drawing Instruments vii

Hints on the Manner of Using Drawing Instruments xi

General Hints on Drawing xii

PRACTICAL PLANE GEOMETRY.

SECTION

I. Lines and Angles 1

II. Triangles 20

III. Quadrilateral Figures 34

IV. Circles, Tangents, and Arcs . . . 44

V. Polygons 56

VI. Ellipses, &c. 73

VII. Inscribed Figures 87

VIII. Described Figures 115

IX. Proportionals 124

X. Similar Figures 131

XI. Equivalent Areas 137

XII. Miscellaneous Problems 167

XIII. Scales 181

PRACTICAL SOLID GEOMETRY.

SECTION		PAGE
I.	Definitions, &c.	187
II.	Projection of Points, Lines, &c.	194
III.	Elementary Solids	199
IV.	Traces of Lines and Planes	212
V.	Further Projections of Solids	226
VI.	Sections	245
VII.	Penetrations of Solids	260

Miscellaneous Exercises in Practical Plane Geometry 264

Miscellaneous Exercises in Practical Solid Geometry 273

Questions in Plane Geometry 277

Questions in Solid Geometry 280

Etymology of Geometrical Terms . . . 282

Index to Problems in Plane Geometry . . . 287

Index to Problems in Solid Geometry . . . 293

INTRODUCTORY SECTION.

ON DRAWING INSTRUMENTS AND THE MANNER OF USING THEM, &c.

For drawing geometrical figures, the most essential instruments are the *compasses* and the *ruler*.

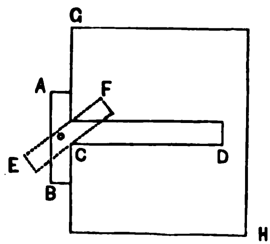
1. **The Compasses.**—These are of various kinds, *e.g.*, we have (1) a pair of *dividers*, having two steel points. These are chiefly used for measuring distances. (2) A pair of *bow-pencil compasses*, having one of its legs furnished with a holder for a pencil. These are used for describing circles. (3) A pair of *bow-pen compasses*, having one of its legs furnished with a mathematical pen. These are used for describing circles in *ink*.

2. **The Ruler.**—These may be of any length, though for the most part they are either 6 inches or a foot. They should have a bevelled edge, and should be divided into inches. They are used for drawing straight lines.

In addition to the foregoing, the student will also require

3. **The Drawing-Board.**—[Its construction is so well known, that any description of it is unnecessary.]

4. **The T Square.**—This instrument consists of two straight

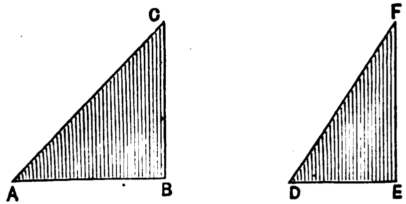


rulers fixed at right angles to each other, as shown in the foregoing diagram. It is used for drawing perpendicular and parallel lines, the cross piece, stock, or hilt, *AB*, being made to slide along the edge of the drawing-board, all straight lines drawn along the edge of the blade, *CD*, will be parallel to one another.

NOTE 1. A shifting bevel piece, *EF*, with clamping screw, is sometimes attached to the hilt of the square, which enables us to draw parallel lines having any given inclination to the sides of the drawing-board or to the base line of the drawing.

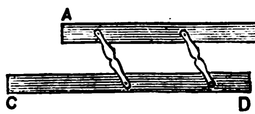
NOTE 2. In the figure, *GH* represents a drawing-board.

5. The Set-Square.—This is a triangular piece of wood *ABC*, having the edge *BC* at right angles to *AB*. It forms a cheap and convenient instrument for drawing perpendicular or parallel lines



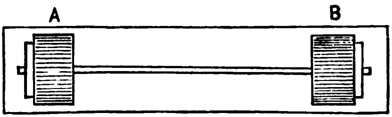
on paper. Set-squares are of *two* kinds, *e.g.*, we have (1) the set-square of 45°; having a *right-angle* at *B*, and each of the angles at *A* and *C* 45°; (2) the set-square of 60° having a *right-angle* at *E*, an angle of 60° at *D*, and hence an angle of 30° at *F*. (**Euc. I. 32.**)

6. The Parallel Ruler.—This is a very simple instrument for drawing parallel lines. It consists of two rulers fixed parallel to each other by means of two equal brass links which are fastened to



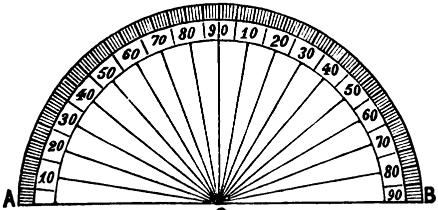
the rulers at equal distances by pivots. The edge, *CD*, of one ruler being placed along a straight line, a pencil mark drawn along the edge, *AB*, of the other ruler will trace a parallel straight line.

7. The Rolling Parallel Ruler.—This instrument consists of a



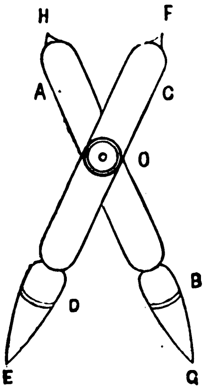
ruler, *AB*, generally divided into inches and tenths; near the extremities are placed two rollers turning on an axis parallel to the edge *AB*, so that the ruler is capable of moving at right angles to the direction of the edge *AB*. By the aid of this instrument, any number of lines may be drawn parallel to a given line, and at any given distance from each other.

8. The Protractor.—This instrument is used for measuring angles, and for laying down angles on paper of any proposed magnitude. It consists of a brass semicircle, *ACB*, the circumference of which is divided into degrees. To lay down any proposed angle, say



60 degrees, draw a line along the edge, *AB*; place a mark coincident with the centre, *C*, and another mark coincident with 60°, as figured on the brass circle, then a line drawn on the paper between these two points will give two straight lines inclined to each other at an angle of 60°.

9. Proportional Compasses.—This instrument is used for reducing or enlarging a figure in any required proportion. It consists



of two brass legs, *AB* and *CD*, terminating at both ends with fine steel points, *E*, *F*, *G*, *H*. The legs turn on the pivot *O*, which may be adjusted so as to divide the length of the legs from point to point in any proportion. Now, whatever the opening of the compasses may be, the distances *GE*, and *HF*, between the points will be in the same proportion to each other as the lengths *OG* and *OH*, into which the legs are divided by the pivot. For example, if *OG* be double *OH*, then *GE* will be double *HF*, and the legs so divided would enable us to enlarge a figure to double its size, or to reduce it to half its size.

10. Diagonal Scale.

11. Scale of Chords.

NOTE. For a description of the last two instruments, see pages 182 and 183 respectively.

HINTS ON THE MANNER OF USING DRAWING INSTRUMENTS.

1. **Pencils.**—For drawing purposes, two of these at least are necessary, viz., the H pencil, and the HHH; one (HHH) for drawing *lines of construction*, the other (H) for the *result lines*. These should not have a needle point, but flattened like a chisel. Great care should be taken in sharpening a pencil, and when it gets short, it should be placed in a crayon-holder.

2. **Compasses.**—These should be always held by the head, otherwise should your fingers touch the sides, the radius might be altered. Great care should also be taken to keep the joints of the compasses *tight*. The extremities of the compasses should also be sharp-pointed, but the paper should be pierced as little as possible by the point which constitutes the centre of the circle. For the *bow-pencil compasses*, "engineers' pencils" are used.

3. **T Square.**—In making use of this instrument, keep to the *same* edge of the drawing-board, over the *same* drawing; otherwise the drawing will probably be inaccurate. Those T squares which have the hilt passing *over* the blade are in practice the most convenient. A T square should also be tested from time to time.

4. **Set-Squares.**—These are useful instruments for setting off angles of 45°, 60°, or 30° as required.

5. **Parallel Ruler.**—In using this instrument, it should be held tight, with two or more fingers of the left hand. As its action is imperfect, for drawings where great exactness is required, it would be better to make use of a set-square and a straight-edge.

GENERAL HINTS ON DRAWING.

THE figures should be first drawn with a black lead pencil. The lines should be as fine as possible, the india-rubber being used sparingly, before the drawing is inked in.

The drawing paper should of course be clean and smooth or hot pressed. Should it be greasy, add a little ox-gall to the ink.

Figures should be drawn on as large a scale as convenient, as the larger the scale, the more correct, generally speaking, is the solution.

Straight lines and arcs should always be drawn sufficiently long at first, as you cannot produce a line, or continue an arc, with such accuracy, if the pencil is taken off the paper, or the point of the compasses is removed from the centre. Both kinds of lines should be smooth, and of uniform thickness.

Lines should be drawn on the surface of the paper, and not indented into it.

When a line is to be drawn parallel to a *short* line, it is better first of all to produce the short line indefinitely both ways, and then proceed.

When several lines pass through *one* point, it is better to commence each line *at* the point which is common to them all.

In inking-in, it is better to take the curve lines before those which are straight.

Intersecting points are best determined when the lines or circles cut one another pependicularly. The point is not so well determined, when the lines or circles cut one another at very acute or very obtuse angles.

A COURSE OF PROBLEMS

IN

PRACTICAL PLANE GEOMETRY.

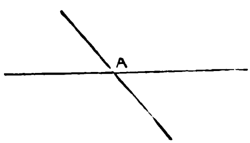
Section I.—LINES AND ANGLES.

DEFINITIONS.

1. A **point** denotes *position* only. It has no magnitude, hence the true mathematical point is merely the *centre* of the dot. *Ex. A*—

2. A **line** has length only, and no breadth, so that it merely indicates *direction*. *Ex. AB*—

The ends of lines are points, and when lines cut each other they are said to *intersect*, and the point where they cross each other is called the *point of intersection*. *Ex. A*—



Note.—Lines are of two kinds—viz., *straight* or *right* lines, and *curved* lines.

3. A **straight line** is the shortest distance between two points. A line is said to be *produced* when it is lengthened at either extremity. *Ex. BC*—

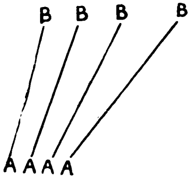
4. A **curved line** is nowhere straight. *Ex. AB*—

NOTE.—The direction of a straight line may be *horizontal*, *vertical*, or *oblique*.

5. A **horizontal line**, as its name implies, is *perfectly level*, like the natural horizon when seen from the midst of the ocean. *Ex. AB*—

6. A **vertical line** is *perfectly upright*, like a plumb-line. *Ex. AB*—

7. An **oblique line** is neither horizontal nor vertical. *Ex. AB*—

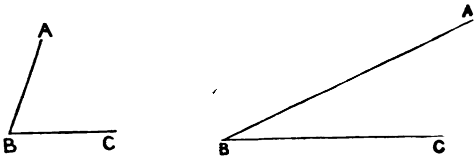


NOTE.—It follows that, while there can be but *one* horizontal line and *one* vertical, the number of oblique lines may be *infinite*.

8. **Parallel lines** are those which are throughout equally distant from each other, and which, therefore, if produced, can never meet. *Ex. AB* and *CD*—

NOTE.—Parallel lines may be *straight* or *curved*.

9. An **angle** is the opening between two straight lines which meet in one point. Its magnitude depends on the *mutual inclination* of the two lines, and not on their lengths. *Ex. ABC*—

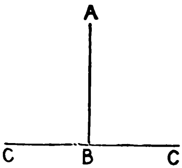


NOTE.—Angles are of three kinds—viz., the *right angle*, the *obtuse angle*, and the *acute angle*.

10. A **perpendicular**. A straight line is said to be perpendicular to another straight line when it stands on it in such a manner that the *adjacent angles* are *equal* to each other. *Ex. AB*—

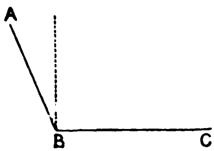
NOTE.—It follows that a perpendicular line is *not* necessarily a vertical line.

11. A **right angle** is the opening between two lines which are perpendicular to each other. *Ex. ABC*—

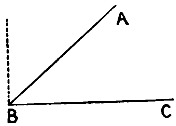


NOTE.—Because the right angle is *invariable* in magnitude, it is made the standard with which all other angles are compared.

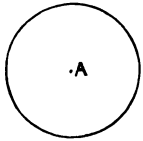
12. An **obtuse angle** is *greater* than a right angle. *Ex. ABC*—



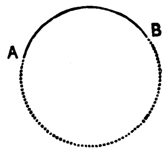
An **acute angle** is *less* than a right angle. *Ex. ABC*—



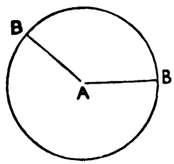
14. A **circle** is a figure contained by one curved line, which is called its *circumference*, and is such that every portion of it is equidistant from a certain point within it called its *centre*. *Ex. A*—



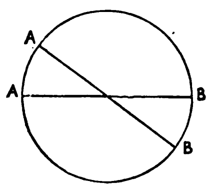
15. An **arc** is any portion of the circumference. *Ex. AB*—



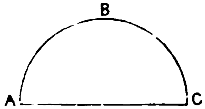
16. A **radius** (plural *radii*) is a straight line drawn from the centre to the circumference. *Ex. AB*—



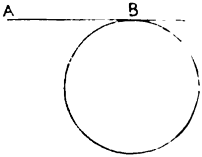
17. A **diameter** is a straight line drawn through the centre, and terminated at both extremities by the circumference. *Ex. AB*—



18. A **semicircle** is half a circle, and it is contained by a diameter and half the circumference. *Ex. ABC*—



19. A **tangent** is a straight line which meets a circle, and, being produced, does not cut it. *Ex. AB.* The point where it *touches* the circle is called the *point of contact.* *Ex. B.*

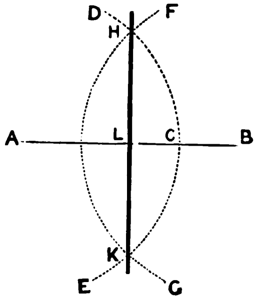


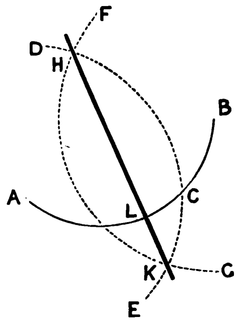
NOTE 1.—The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees*, marked °. Hence, a semicircle will contain 180°; a quarter of a circle, or a *quadrant*, 90°; the sixth part, 60°, &c.

NOTE 2.—Angles at the centre of a circle are proportional to the arcs on which they stand. Hence, a quadrant will contain an angle of 90°, *i.e.*, a right angle. A degree is divided into 60 equal parts, called *minutes*, marked ′; and each minute into 60 equal parts, called *seconds*, marked ″. Thus, 44° 30′ 27″ reads—44 *degrees*, 30 *minutes*, 27 *seconds*.

Problem 1.

To **bisect** *a given* **straight line** *AB, or a given* **arc** *AB; that is, to divide it into two equal parts.*





1. From point *A* as centre, with any radius *greater than half the line AB*, describe the arc *DE*.

2. From point *B* as centre, *with the same radius*, describe the arc *FG*, intersecting the arc *DE* in *H* and *K*.

3. Draw the straight line *HK*, and *the given straight line AB will be bisected in the point L.*

NOTE 1.—*HK* is perpendicular to *AB*, and at right angles to it.

NOTE 2.—The same method is to be followed in bisecting the given arc *AB*.

Problem 2.—(A.)

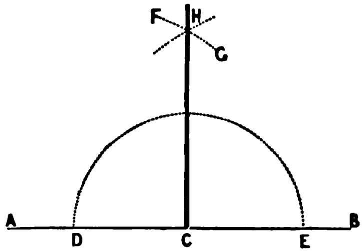
To draw a straight line **perpendicular** *to a given* **straight line** *AB, from a given point in the line.*

First. Let the given point *C* be at or near the **middle** of the line *AB*.

1. From point *C* as centre, *with any convenient radius*, describe a semicircle meeting *AB* in points *D* and *E*.

2. From point D as centre, ***with any radius***, describe arc FG; and from E as centre, ***with the same radius***, intersect the arc FG in the point H.

3, Draw the straight line HC, and ***it will be perpendicular the given straight line AB.***

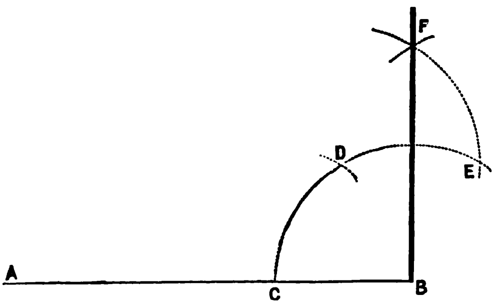


NOTE 1.—Because HC is perpendicular to AB, each of the angles ACH, BCH is a right angle.

NOTE 2.—In naming an angle, the *middle* letter should be at the angle.

(B.)

Secondly. Let the given point B be at or near one **end** of the line AB.



1. From point B as centre, ***with any convenient radius*** describe an arc CDE.

2. From point *C*, *with the same radius*, cut the arc in *D*; from *D*, *with the same radius*, describe an arc *EF*, cutting *CDE* in *E*; and from *E*, *with the same radius*, cut the arc *EF* in *F*.

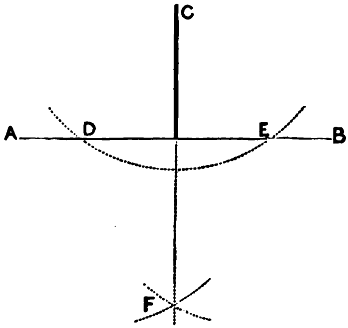
3. Draw the line *FB*, and *it will be perpendicular to, or at right angles to, the given straight line AB.*

Problem 3.—(A.)

To draw a straight line **perpendicular** *to a given* **straight line** *AB, from a given point outside it.*

First. Let the given point *C* be opposite, or nearly opposite, the **middle** of the line *AB*.

1. From point *C*, *with any sufficient radius*, describe an arc cutting *AB* in *D* and *E*.



2. From points *D* and *E* as centres, *with any radius*, describe arcs cutting each other in the point *F*.

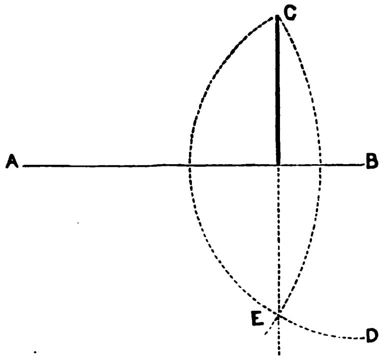
3. Draw the line *CF*, and *it will be perpendicular to the given straight line AB.*

(B.)

Secondly. Let the given point *C* be opposite, or nearly opposite, one **end** of the line *AB*.

1. From point *B* as centre, *with radius BC*, describe arc *CD*.

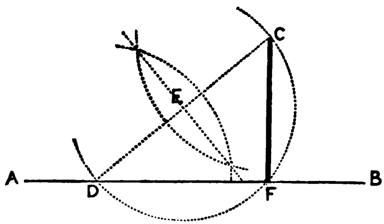
2. From point *A* as centre, *with radius AC*, describe arc *CE*.



3. Draw the line *CE*, *and it will be perpendicular to the given straight line AB.*

Another Method.

1. Take *any* point *D* in the given line *AB* towards *A*. Join *DC*, and bisect it in *E* (**Pr. 1**).



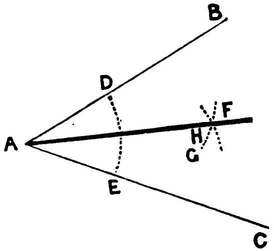
2. From *E* as centre, *with ED as radius*, describe an arc cutting *AB* in *F*.

3. Draw the line *CF*, and *it will be perpendicular to the given straight line AB.*

Problem 4.

To **bisect** *a given* **angle** *BAC.*

1. From point *A* as centre, *with any radius*, describe an arc *DE*.

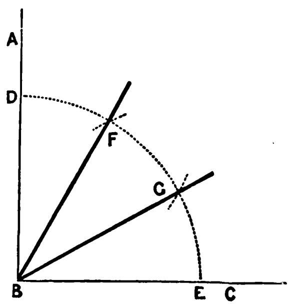


2. From point *D* as centre, *with any radius*, describe the arc *FG;* and from point *E* as centre, *with the same radius*, intersect the arc *FG* in the point *H*.

3. Draw the straight line *AH*, and *it will bisect the given angle BAC.*

Problem 5.

To **trisect** *a given* **right angle** *ABC, that is, to divide it into three equal angles.*



1. From point *B* as centre, *with any radius*, describe an arc *DE*.

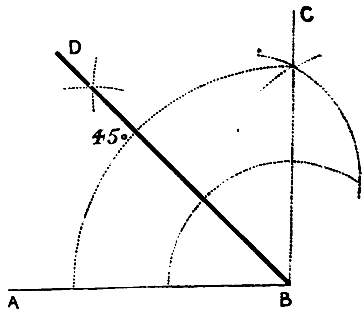
2. From centres *D* and *E*, *with the same radius*, cut arc *DE* in *G* and *F*.
3. Draw the straight lines *BF* and *BG*, and *the given right angle ABC will be trisected, that is, divided into three equal angles.*

NOTE.—It is only the *right* angle which (strictly speaking) can be *trisected* by a plane geometrical construction.

Problem 6.

To construct an **angle** *of* **45°** *at point B on a given* **line** *AB.*

1. At point *B*, erect a perpendicular to *AB* (**Pr. 2**). *ABC* will then be a right angle, that is, an angle of 90°.



2. Bisect the right angle *ABC* (**Pr. 4**) by the line *BD*. *Then DBA is the required angle.*

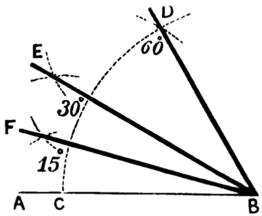
NOTE.—In the same manner, an angle of 22½° may be constructed by bisecting the angle *DBA*.

Problem 7.

To construct an **angle** *of* **60°**, **30°**, *or* **15°** *at point B on a given* **line** *AB.*

1. With centre *B*, *and any radius*, describe an arc cutting *AB* in *C*.
2. From centre *C*, *with the same radius*, cut the arc in *D*.
3. Draw the line *DB*. *Then DBA is the required angle of* 60.

4. Bisect the angle DBA by the line EB (**Pr. 4.**) *Then EBA is the required angle of* 30°.
5. Bisect the angle EBA by the line FB (**Pr. 4**). *Then FBA is the required angle of* 15°.



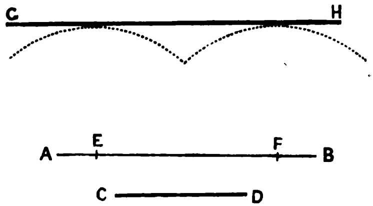
NOTE 1.—In the same manner, an angle of $7\frac{1}{2}°$ may be constructed by bisecting the angle FBA.

NOTE 2.—The *radius* of a circle is *one-sixth* of its circumference. It is on this principle that the angle DBA is 60°.

NOTE 3.—By means of this problem, we might construct *other* angles. Thus, if DBA be trisected by drawing lines to B, we obtain an angle of 20°. Bisect that, and we obtain an angle of 10°. Again, if angle FBC be trisected, by drawing lines to B, we might obtain an angle of 5°.

Problem 8.

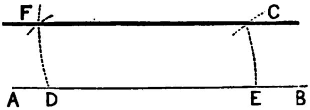
To draw a line **parallel** *to a given* **line** *AB, at a given* **distance** *from it, as CD.*



1. Take *any* two points, E and F in the line AB, and with E and F as centres, *and radius CD,* describe arcs above the line.
2. Draw the line GH *tangential* to, or *touching* the arcs. *Then the line GH will be parallel to the given line AB, and at the given distance CD from it.*

Problem 9.

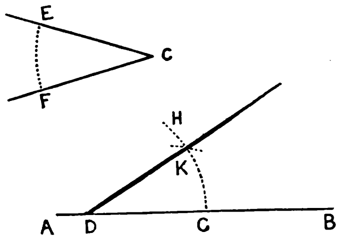
To draw a line **parallel** *to a given* **line** *AB, through a given* **point** *C.*



1. Take *any* point *D* in the given line *AB* towards *A* as centre ; and from *D, with radius DC,* describe an arc cutting *AB* in *E.*
2. From *C* as centre, *with the same radius,* describe an arc *DF,* and make arc *DF* equal to arc *CE.*
3. Draw the line *FC,* and *it will be parallel to the given line AB.*

Problem 10.

From a given **point** *D, in the line AB, to make an* **angle** *equal to the given* **angle** *C.*

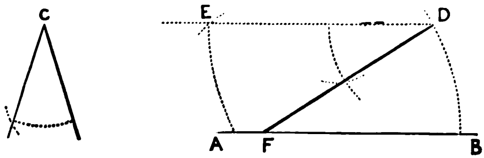


1. From point *C* as centre, *with any convenient radius,* describe an arc *EF.*
2. From point *D, with the same radius,* describe the arc *GH.*
3. From point *G, and the distance EF,* cut off *GK* equal to *EF.*
4. Draw the line *DK,* and *the angle KDB will be equal to the given angle C.*

Problem 11.

To draw a **line** *from a given* **point** *D, outside a given line AB, making with the given line, an* **angle** *equal to* a *given* **angle** *C.*

1. From the point *D* draw a line *DE* parallel to *AB* (**Pr. 9**).

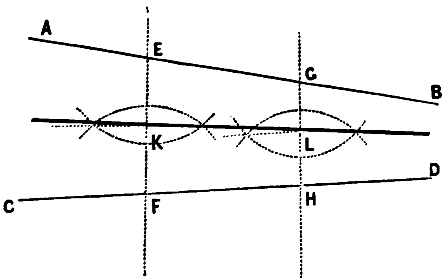


2. At the point *D* make an angle *EDF* equal to the angle *C* (**Pr. 10**).
3. Produce *DF* to meet *AB* in *F*. *Then the angle DFB will be equal to the angle EDF, which is equal to the given angle C* (constr.)

NOTE.—We know from Euclid I. 29, that "if a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another," hence angle *EDF* is equal to angle *DFB.*

Problem 12.

To **bisect** *the* **angle** *made by any two given* **converging lines** *AB and CD, when the* **angular point** *is* **inaccessible.**

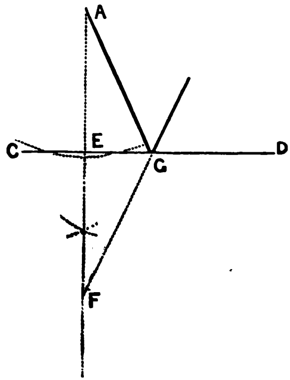


1. Draw *any* two parallel lines *EF* and *GH,* across *AB* and *CD* (**Pr. 8**).

2. Bisect EF and GH in the points K and L respectively (**Pr. 1**).
3. Draw a line through K and L, the points of bisection. *This line, if produced, will bisect the angle made by the produced given lines AB, CD.*

Problem 13.

To draw **straight lines** *from any two given* **points** *A and B, outside a given straight line CD, so as to make* **equal angles** *with the given* **line** *CD.*



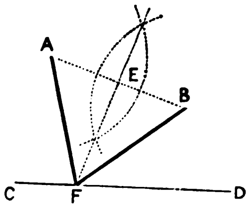
1. From A, let fall a perpendicular AE on CD (**Pr. 3**), and produce it *indefinitely* towards F.
2. Make EF equal to AE, and join BF, cutting CD in G.
3. Draw the line AG. *Then the angles AGC and BGD are equal, and AG and BG are the required lines.*

Problem 14.

To draw **straight lines** *from any two given* **points** *A and B outside a given straight line CD, and to meet CD, so that they may be* **equal** *in length.*

1. Draw the straight line AB, and bisect it in E (**Pr. 1**).

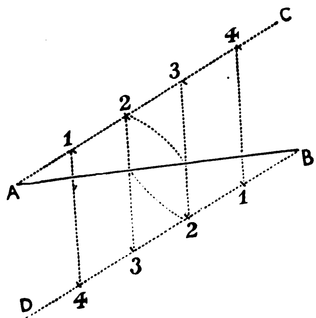
2. Produce the bisecting line to meet *CD* in *F*.



3. Join *AF* and *BF*, *which will be the two required lines.*

Problem 15.

To **divide** *a given* **line** *AB into any number of* **equal** *parts (say in this case* **five***).*



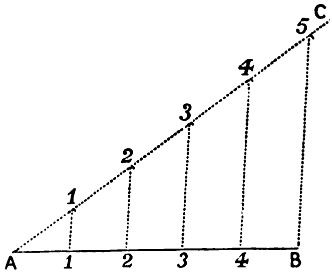
1. Draw a line *AC* at *any* angle with *AB*, and draw *BD*, making the angle *ABD* equal to the angle at *A* (**Pr. 10**).

2. Commencing at *A*, mark off on *AC* the number of points, *less one*, that *AB* is to be divided into, *i.e.*, set off *four* equal parts of *any* length, as 1, 2, 3, 4.

3. From *B*, mark off on *BD* the *same* number of equal parts, as 1, 2, 3, 4.

4. Join 1 4, 2 3, 3 2, &c., and *the given line AB will be divided into five equal parts.*

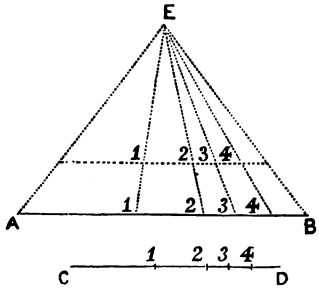
Another Method.



1. Draw a line *AC* of *any* length, and making *any* angle with *AB*.
2. From *A* mark off *any* five spaces on *AC*.
3. From the end of the last equal space, draw a line to *B*, as 5*B*.
4. From the remaining points of division between *A* and 5, draw lines to *AB*, but parallel to 5*B*, as 4 4, 3 3, &c., *then the given straight line AB will be divided into five equal parts.*

Problem 16.

To **divide** *any* **line** *AB* **proportionally** *to a given* **divided line** *CD.*

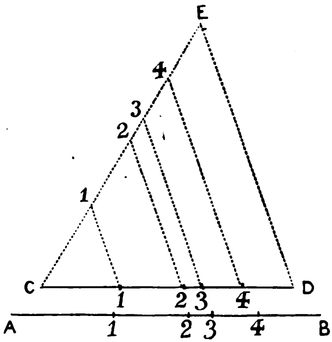


1. Draw a line *similarly* divided to *CD*, *parallel* to *AB*, and at *any* distance from it (**Pr. 8**).

2. From A and B, draw AE and BE *through the ends of this line* to meet in E.

3. Draw lines from E, through the points of division, 1, 2, 3, 4, cutting AB in 1, 2, 3, 4, *then AB is divided proportionally to the given divided line CD.*

Another Method.



1. Draw CE equal to AB, and at *any* angle with CD, and join ED.
2. Draw 4 4, 3 3, &c., *parallel* to ED.
3. Transfer the divisions on the line CE to the given line AB, and then *the divisions on the given line AB will bear the same proportion to each other that the divisions on CE bear to each other.*

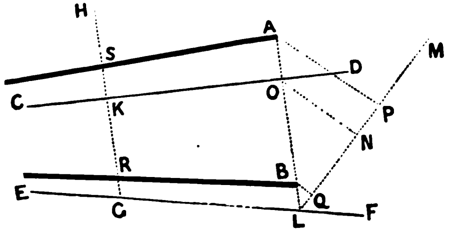
Problem 17.

To draw lines from any two given **points** *A and B, which shall go to the same point to which any two given* **lines** *CD and EF converge, when the* **point of convergence** *is* **inaccessible.**

1. From A, through B, draw AL. From any point G in the line EF, draw GH of unlimited length, parallel to AL (**Pr. 9**), cutting CD in K.

2. From L, draw LM of unlimited length, and at *any* angle.

3. Mark off *LN* equal to *GK*. Join *ON*, and from *A* and *B*, draw *AP*, *BQ* parallel to *ON* (**Pr. 9**). The divisions *L*, *Q*, *N*, *P*, are *proportional* to the divisions *L*, *B*, *O*, *A* (**Pr. 16**).



4. Make *GR* equal to *LQ*, and *KS* equal to *NP*. Then the divisions from *G* to *S* are in the *same proportion* as the divisions from *L* to *A*. Hence, *if we draw the lines AS, BR, they will converge to the same point as the given lines CD, EF.*

Section II.—TRIANGLES.

DEFINITIONS.

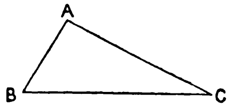
(*Hitherto we have treated only of* LINES *and* ANGLES.)

1. "A **figure** is that which is inclosed by one or more boundaries" (**Euc. I. Def. 14**).

2. "**Rectilineal figures** are those which are contained by straight lines" (**Euc. I. Def. 20**).

   NOTE.—"*Two* straight lines cannot inclose a space" (**Euc. I. Ax. 10**). Hence the triangle is the most simple of all rectilineal figures.

3. A **triangle** is a figure which is bounded by three straight lines. *Ex. ABC*—

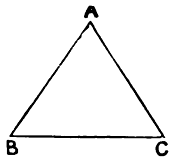


   NOTE 1.—It is therefore called a trilateral, or *three-sided* figure.

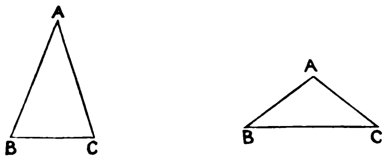
   NOTE 2. As every rectilineal figure contains as many angles as sides, it is often named from its anglés or *corners.* Hence the term *triangle.*

   NOTE 3.—Triangles are of *six* kinds. *Three* of these are named from the comparative lengths of their *sides,* and *three* from the sizes of their *angles.*

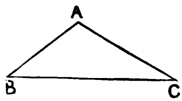
4. An **equilateral triangle** is that which has three equal sides. *Ex. ABC*—



5. An **isosceles triangle** is that which has only two sides equal. *Ex. ABC*—

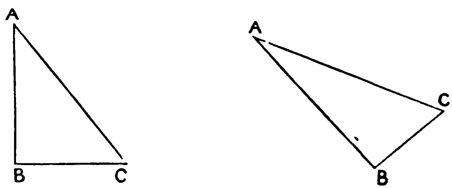


6. A **scalene triangle** is that which has three unequal sides. *Ex. ABC*—



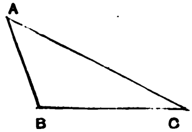
NOTE.—The above terms have reference to the *sides* of the triangle.

7. A **right-angled triangle** is that which has a right angle. *Ex. ABC*—

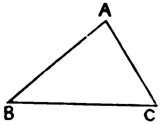


NOTE.—The side which is opposite the right angle (*AC*) is called the *hypotenuse*. The other sides are called the *base*, and *perpendicular*, irrespective of the position of the figure.

8. An **obtuse-angled triangle** is that which has one of its angles an obtuse angle. *Ex. ABC*—

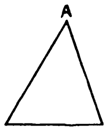


9. An **acute-angled triangle** is that which has three acute angles. *Ex. ABC*—



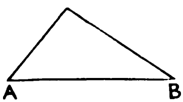
NOTE.—The above terms have reference to the *angles* of the triangle.

10. The **vertex** (plural *vertices*) of a triangle is its *highest* angle. *Ex. A*—

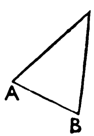


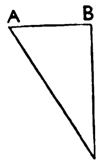
NOTE.—It is also called the *apex*, or the *vertical angle.*

11. The **base** of a triangle is *generally* its *lowest* side. *Ex. AB*—



NOTE.—Both in an isosceles triangle and in a right-angled triangle, the position of the base is changed.

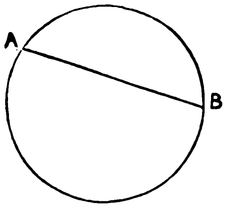




12. The **altitude** of a triangle is its perpendicular height, *i.e.*, the length of a perpendicular drawn from the vertex to the base, or to the base produced. *Ex. AB*—

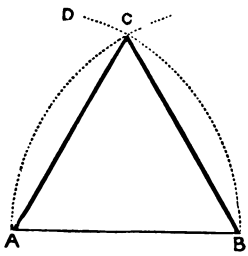
13. The **perimeter** of a figure is its *whole* boundary. *Thus, if one side of an equilateral triangle be* 5, *its perimeter is* 15.

14. A **chord** is *any* straight line drawn across a circle, provided that it does *not pass through the centre. Ex. AB*—



Problem 18.

To construct an **equilateral triangle** *on a given* **base** *AB.*



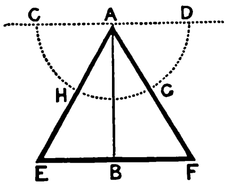
1. From centre *A*, *with radius AB*, describe an arc *BD*.

2. From centre *B*, *with the same radius*, cut the arc in *C*.
3. Join *AC* and *BC*. *Then ABC is the required equilateral triangle.*

NOTE.—The three straight lines are all equal, since they are *radii* of equal arcs.

Problem 19.

To construct an **equilateral triangle** *having a given* **height** *AB*—



1. From the extremities of the line *AB*, draw *CAD* and *EBF* perpendicular to it (**Pr. 2**).
2. From *A* as centre *with any radius*, describe a semicircle cutting *CAD* in *C* and *D*.
3. From *C* and *D with the same radius*, cut the semicircle in *G* and *H*.
4. From *A* draw lines through *H* and *G*, meeting *EF* in *E* and *F*. *Then AEF is the equilateral triangle required.*

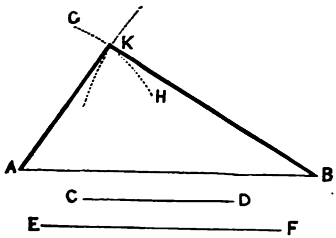
NOTE.—The radius of a circle can be marked off *six* times round its circumference, hence the arc *HG* is 60°. Moreover, the three angles of a triangle, added together, are equal to two right angles, or 180° (**Euc. I. 32**). Hence, the angle *HAB* being 30°, and *ABE* 90°, the angle *AEB* is 60°.

Problem 20.

To construct a **triangle,** *the* **three sides** *AB, CD, and EF being given.*

1. With centre *A*, *and radius CD*, describe arc *GH*.
2. With centre *B*, *and radius EF*, describe an arc cutting *GH* in *K*.

3. Draw the straight lines *AK*, *BK*, *then KAB is the triangle required.*

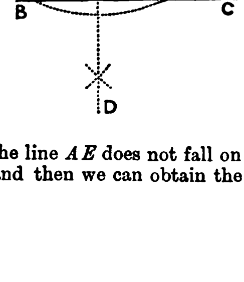


NOTE.—"The greater side of every triangle is opposite to the greater angle" (Euc. I, 18). Hence angle *AKB* is greater than the angle *KAB*.

Problem 21.

To find the **altitude** *of a given* **triangle** *ABC.*

1. From point *A*, let fall the perpendicular *AD* (**Pr. 3**).
2. *Line AE is the required altitude of the given triangle ABC.*

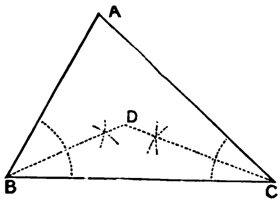


NOTE.—If the line *AE* does not fall on the base, the base must be *produced*, and then we can obtain the altitude of the triangle as above.

Problem 22.

To find the **centre** *of a given* **triangle** *ABC.*

1. Bisect any two of its angles, say, the angles at *B* and *C* (**Pr. 4**).
2. Produce the bisecting lines, and let them meet in *D*. *Then D is the centre of the given triangle ABC.*

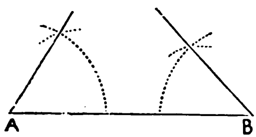


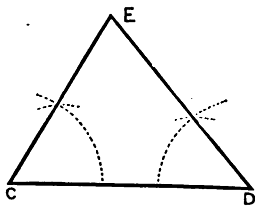
NOTE.—Perpendiculars drawn from *D* to the three sides of the triangle are equal in length. They would thus become the *radii* of a circle which might be inscribed within the triangle (**Euc. IV. 4.**)

Problem 23.

To construct a **triangle,** *its* **base** *AB and the* **angles at the base** *A and B being given.*

1. Draw line *CD* equal to *AB.*





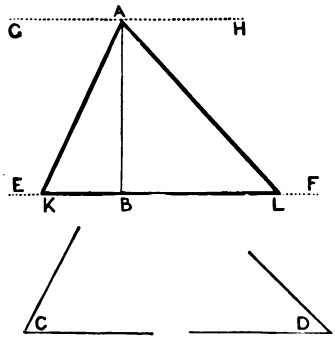
2. Make angle *C* equal to angle *A*, and angle *D* equal to angle *B* (**Pr. 10**).
3. Produce the sides until they meet in *E*. *Then CED is the required triangle.*

Problem 24.

To construct a **triangle,** *the* **altitude** *AB, and the two* **angles at the base,** *C and D, being given.*

1. Through the point *B*, draw *EF* perpendicular to *AB* (**Pr. 2**); also through *A*, draw *GH* perpendicular to *AB*.

2. From point *A* draw *AK*, making the angle *AKB* equal to angle *C*, by *first* making angle *GAK* equal to *C* (**Pr. 10**).



3. In the same manner, make angle *ALB* equal to *D*. *Then AKL is the required triangle.*

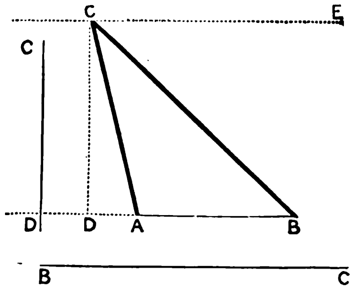
NOTE.—The angles *GAK*, *AKB*, are called *alternate* angles, and when a straight line falls upon two parallel straight lines, it makes the alternate angles equal to each other (**Euc. I. 29**).

Problem 25.

To construct a **triangle,** *having its* **base** *AB, its* **altitude** *CD, and one* **side** *BC given.*

1. Draw a line *CE* parallel to *AB*, at a distance from it equal to the altitude *CD* (**Pr. 8**).

2. From B, as centre, *with radius* BC, cut CE in the point C.

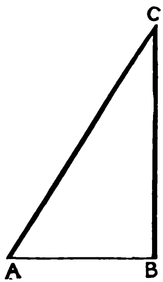


3. Join CB, CA. *Then* ABC *is the required triangle, and* CD *drawn from* C *perpendicular to the base* AB *produced* (**Pr. 3**) *is the altitude.*

Problem 26.

To construct a **triangle** *on a given* **base** AB, *having angles of* **60°**, **30°**, *and* **90°**.

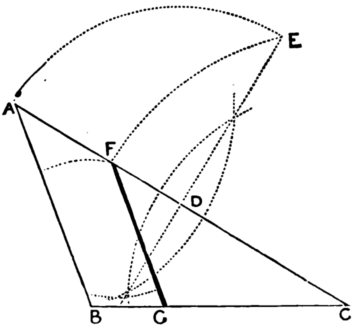
1. At B, construct a right angle, that is, raise a perpendicular (**Pr. 2**).



2. At A, make an angle of 60° (**Pr. 7**), and continue the line, until it meets the perpendicular erected at B, in the point C. *Then* ABC *is the required triangle.*

Problem 27.

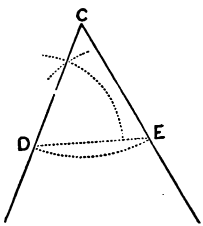
To **bisect** *any given* **triangle** *ABC by a* **line** *drawn* **parallel** *to one of its* **sides** *AB.*

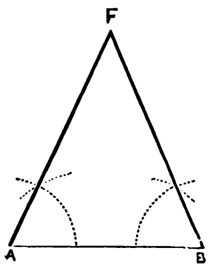


1. Bisect one side as *AC* in *D* (**Pr. 1**), produce the bisecting line towards *E*, and make *DE* equal to *DA* or *DC*.
2. From *C*, *with radius CE*, describe an arc meeting *AC* in *F*.
3. From *F*, draw a line *FG* parallel to *AB* (**Pr. 9**), and meeting *BC* in *G*. *Then the given triangle ABC will be bisected by the line FG.*

Problem 28.

To construct an **isosceles triangle** *on a given* **base** *AB, and having a given* **vertical angle** *C.*





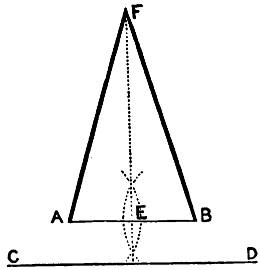
1. From the angular point *C* as centre, *and with any radius*, cut the sides of the angle in *D* and *E*, and join *DE*.

2. At points *A* and *B*, make angles equal to the angles at *D* and *E* respectively (**Pr. 10**).

3. Produce the lines forming the angles to meet in *F*, and *FAB will be the required isosceles triangle.*

Problem 29.

To construct an **isosceles triangle** *having its* **base** *AB and its* **altitude** *CD given.*

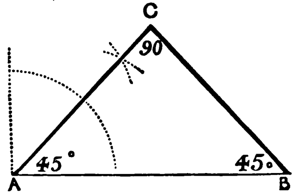


1. Bisect the base *AB* in *E* (**Pr. 1**), and from the point of bisection, *E*, mark off the given altitude *CD* on the line in the point *F*.

2. Join *FA* and *FB*. *Then FAB is the required isosceles triangle.*

Problem 30.

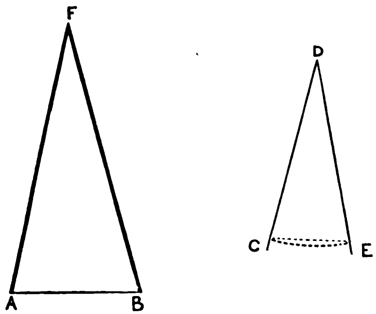
To construct an **isosceles triangle** *on a given* **base** *AB, having a* **vertical angle** *of* **90°**.



1. At point *A*, make a right angle (**Pr. 2**), bisect it (**Pr. 6**), thereby making *BAC* equal to 45°.
2. At point *B*, make an angle *ABC* equal to angle *BAC* (**Pr. 10**). *Then ABC will be the isosceles triangle required, and having its vertical angle ACB of* 90°.

Problem 31.

To construct an **isosceles triangle** *on a given* **base** *AB, its* **vertical angle** *containing a given required number of degrees (say in this case* $22\frac{1}{2}$°*)*.



1. Construct an angle *CDE* containing the required number of degrees, viz. $22\frac{1}{2}$ (**Pr. 6**), and draw a chord to the arc *CE*.

   At points *A* and *B* in the given line *AB*, construct angles equal to that at *C* or *E* (**Pr. 10**).

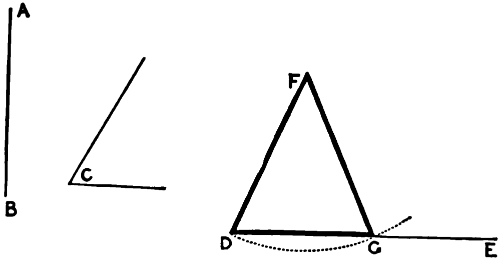
3. Produce the lines completing the angles from *A* and *B* until they meet in *F*. *Then FAB is the required isosceles triangle.*

Problem 32.

To construct an **isosceles triangle, one** *of the equal* **sides** *AB, and* **one** *of the* **equal angles** *C, being given.*

1. Draw *DE* of unlimited length.

2. Make angle *EDF* equal to angle *C* (**Pr. 10**), and make *DF* equal to *AB*.

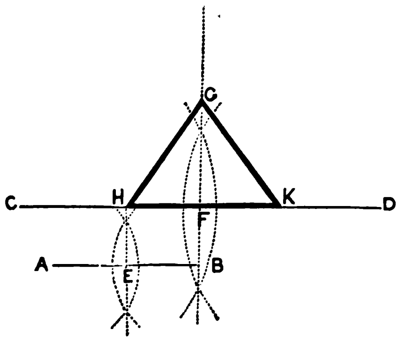


3. With centre *F*, *and radius FD*, describe arc *DG*.

4. Join *FG*, and *FDG is the required isosceles triangle, having its two sides FD, FG equal to the given line AB, and its two angles D, G, equal to the given angle C.*

Problem 33.

To construct an **isosceles triangle,** *having its* **base** *AB and its* **perimeter** *CD given.*



1. Bisect *AB* and *CD* in the points *E* and *F* (**Pr. 1**), and produce the bisecting line through *F* indefinitely towards *G*.

2. From F, *with radius half* AB, that is, AE, cut the line CD in points H and K.

3. From H, *with radius* HC, cut HG in the point G.

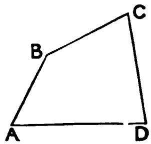
4. Join GH, GK. *Then* GHK *is the isosceles triangle.*

Section III.

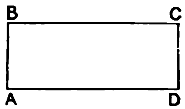
QUADRILATERAL FIGURES.

DEFINITIONS.

1. A **quadrilateral** figure is one which is bounded by four straight lines. *Ex. ABCD*—

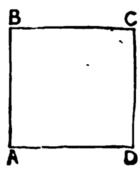


2. A **parallelogram** is a quadrilateral, of which the *opposite* sides are parallel and equal. *Ex. ABCD*—

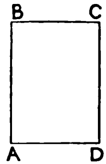


Note.—Quadrilaterals fall into *six* classes, *four* of which are *parallelograms*—viz., the *square*, the *rectangle*, the *rhombus*, and the *rhomboid*.

3. A **square** is a parallelogram which has all its sides equal, and all its angles right angles. *Ex. ABCD.*

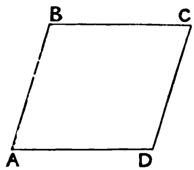


4. A **rectangle** is a parallelogram which has *only* its *opposite* sides equal, but all its angles right angles. *Ex. ABCD*—



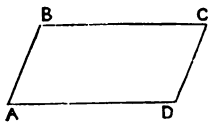
NOTE.—This kind of parallelogram is also termed an *oblong.*

5. A **rhombus** is a parallelogram which has all its sides equal, but its angles are *not* right angles. *Ex. ABCD*—



NOTE.—In each case its *opposite* angles are equal to each other.

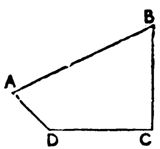
6. A **rhomboid** is a parallelogram which has *only* its *opposite* sides equal, but its angles are *not* right angles. *Ex. ABCD*—



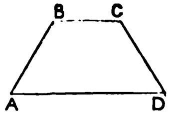
NOTE 1.—As in the preceding figure, its *opposite* angles are equal to each other.

NOTE 2.—The remaining classes of quadrilaterals are the *trapezium* and the *trapezoid.*

7. A **trapezium** is a quadrilateral which has *none* of its sides *parallel. Ex. ABCD*—



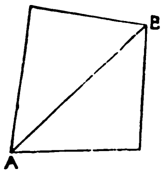
8. A **trapezoid** is a quadrilateral which has only *two* of its sides parallel. *Ex. ABCD*—



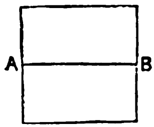
NOTE 1.—Some of its sides and angles *may* be *equal.*

NOTE 2.—*All* quadrilateral figures are also called *quadrangles,* as they have also four angles.

9. A **diagonal** of a quadrilateral is a straight line which joins *any* two of its opposite angles. *Ex. AB*—



10. A **diameter** is a straight line drawn through its centre *parallel* to two of its sides. *Ex. AB*—

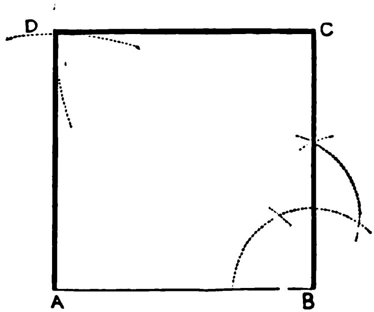


Problem 34.

To construct a **square** *on a given* **base** *AB*—

1. At *B* in the given line *AB* erect a perpendicular *BC* equal to *AB* (**Pr. 2**).
2. From the point *A*, *with radius AB*, describe an arc above *A*.

3. From *C, with the same radius,* cut the arc in *D.*

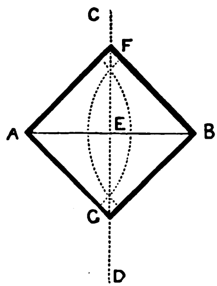


4. Join *AD* and *CD.* *Then ABCD is the square required.*

Problem 35.

To construct a **square,** *the* **diagonal** *AB being given.*

1. Bisect *AB* by the perpendicular *CD* **(Pr. 1).**

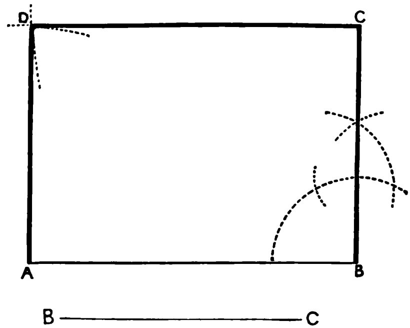


2. Cut off *EF,* and *EG,* equal to *EA* or *EB.*
3. Join *AF, FB, BG, GA.* *Then AFBG is the square required, having the given diagonal AB.*

Problem 36.

To construct a **rectangle,** *the lengths of two of the* **sides** *AB and BC being given.*

1. From the point *B* in the given line *AB*, erect *BC* perpendicular to *AB* (**Pr. 2**), and equal to the given line *BC.*

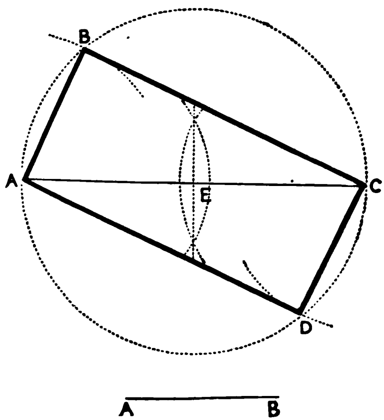


2. From *A*, *with radius BC,* describe an arc above *A.*
3. From *C*, *with radius AB,* cut the arc in the point *D.*
4. Join *AD* and *CD.* *Then ABCD is the required rectangle.*

Problem 37.

To construct a **rectangle,** *one* **side** *AB and its* **diagonal** *AC being given.*

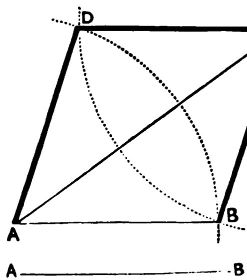
1. Bisect the diagonal *AC* in *E* (**Pr. 1**), and from *E* as centre, *with the radius EA* or *EC*, describe a circle.
2. From *A* and *C* as centres, *with the given line AB as radius,* describe arcs *B* and *D.*
3. Join *AB*, *BC*, *CD*, and *DA.* *Then ABCD is the rectangle required.*



NOTE.—Angle *ABC* is a *right angle,* and the lines drawn from *A* and *C* to *any* point in the arc of the semicircle would form a right angle (Euc. III., 31).

Problem 38.

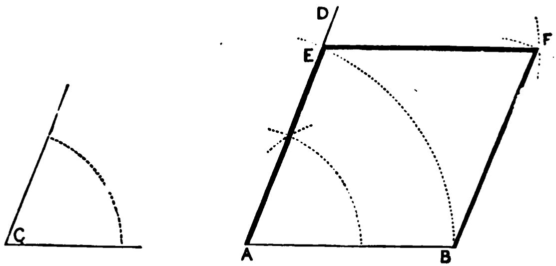
To construct a **rhombus,** *its* **side** *AB and one* **diagonal** *AC being given.*



1. With centres *A* and *C, and radius AB,* describe arcs cutting at *D* and *B*.
2. Join *AD, DC, CB,* and *BA*. *Then ADCB is the rhombus required.*

Problem 39.

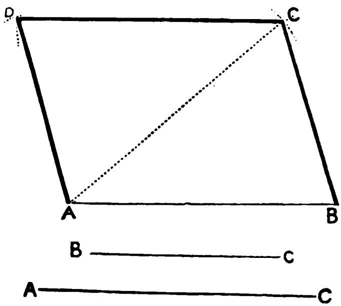
To construct a **rhombus,** *its* **side** *AB and an* **angle** *C being given.*



1. Make angle *BAD* equal to *C* (**Pr. 10**), and cut off *AE* equal to *AB.*
2. With centres *E* and *B, and radius AB,* describe arcs intercepting in the point *F.*
3. Join *EF, FB. Then AEFB is the required rhombus.*

Problem 40.

To construct a **rhomboid,** *its two adjacent* **sides** *AB and BC, and a* **diagonal** *AC, being given.*



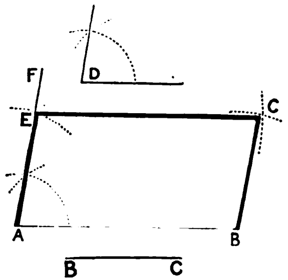
1. From *A* in *AB, with the diagonal AC as radius,* describe an arc.

2. From *B, with radius BC,* cut the arc in *C.*
3. From *C, with AB as radius,* describe an arc.
4. From *A, with radius BC,* cut the arc in *D.*
5. Join *BC, CD,* and *DA.* *Then ABCD is the rhomboid required.*

Problem 41.

To construct a **rhomboid,** *its two adjacent* **sides** *AB and BC, and* **angle** *D being given.*

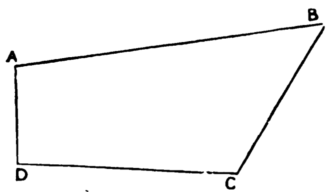
1. At *A* in *AB* make the angle *BAF* equal to the given angle *D* (**Pr. 10**), and cut off *AE* equal to *BC.*



2. From *B, with radius BC,* describe an arc above *B.*
3. From *E* as centre, *with radius AB,* cut the arc in *C.*
4. Join *EC* and *CB.* *Then AECB is the required rhomboid.*

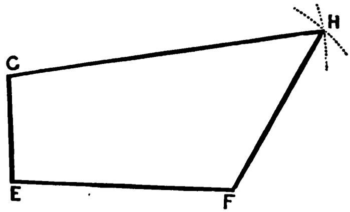
Problem 42.

To construct a **trapezium** *equal to a given* **trapezium** *ABCD.*



1. Make line *EF* equal to *CD,* and the angle at *E* equal to the angle at *D* (**Pr. 10**).

2. Make the side *EG* equal to *AD*.

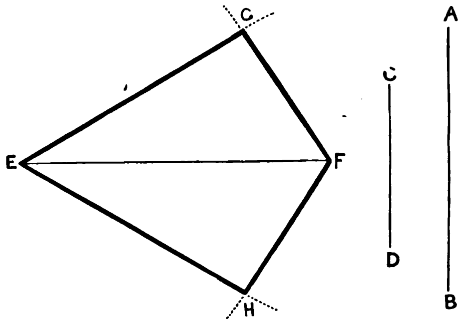


3. From the point *G, with radius AB*, and from *F, with radius BC*, describe arcs cutting in *H*.

4. Join *GH, FH. Then the trapezium EFGH shall be equal to the given trapezium ABCD.*

Problem 43.

To construct a **trapezium**, *having its adjacent pairs of* **sides** *equal respectively to two given lines AB and CD, and its* **diagonal** *equal to the given line EF.*



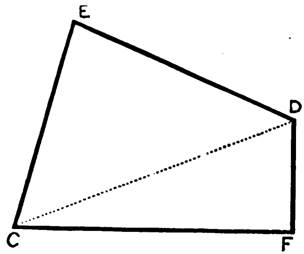
1. From centre *E, with AB radius*, and from centre *F, with CD radius*, describe arcs cutting in *G* and *H*.

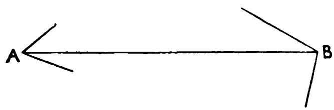
2. Join *EG, GF, FH*, and *HE. Then EGFH is the required trapezium.*

Problem 44.

To construct a **trapezium** *when the length of the* **diagonal** *AB, and the* **angles** *at its extremities A and B are given.*

1. Make any straight line *CD* equal to the diagonal *AB*.





2. Make angles at *C* equal and correspondent to the angles at *A* ; and angles at *D* equal and correspondent to the angles at *B* (**Pr. 10**).

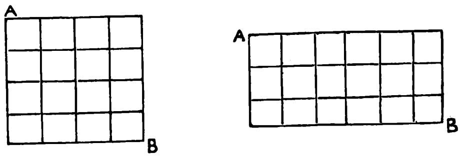
3. Produce their sides until they meet. *Then the figure CEDF is the required trapezium.*

Section IV.—CIRCLES, TANGENTS, AND ARCS.

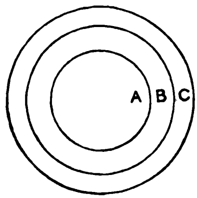
DEFINITIONS.

1. The **area** of a figure is its *superficies* or *surface*. Such measurements are calculated by square or superficial measure. Thus

   (*a*) A *square* whose side is 4 linear inches, contains an area of 16 square inches. *Ex. AB*—



   (*b*) A *rectangle* whose adjacent sides are 6 and 3 linear feet, contains an area of 18 square feet. *Ex. AB*—

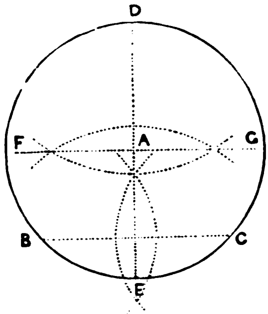


2. **Concentric** circles. Circles are said to be *concentric* when they have a *common* centre. *Ex. A, B, C*—

Problem 45.

To find the **centre** *of a given* **circle** *A.*

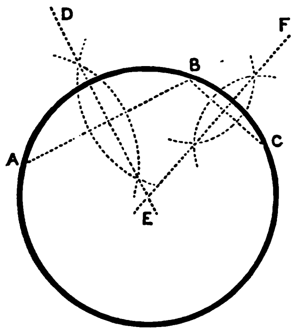
1. Draw any chord *BC*, and bisect it by a line meeting the circumference in *D* and *E* (**Pr. 1**).



2. Bisect *ED* by the line *FG*. *The point of intersection, A, is the centre of the given circle.*

Problem 46.

To describe a **circle** *which shall pass through* **three given points** *A, B, and C.*



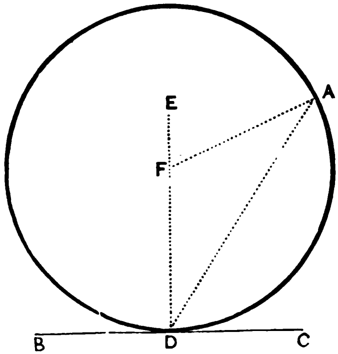
1. Join *AB*, and bisect it by the perpendicular *DE* (**Pr. 1.**)

2. Join *BC*, and bisect it by the perpendicular *FE*, intersecting *DE* in *E*.
3. With centre *E*, *and radius EA*, describe the required circle, *and it will pass through the three given points A, B, C.*

Problem 47.

To describe a **circle** *which shall pass through any given* **point** *A, and which shall also be* **tangential** *to the given* **line** *BC, in a given point D.*

1. Draw *DE* at right angles to *BC* (**Pr.** 2).



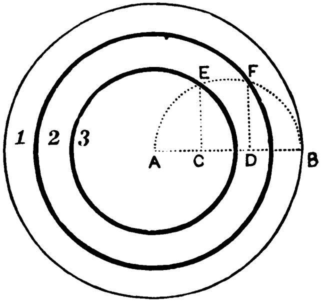
2. Join *AD*, and make the angle *DAF* equal to the angle *ADE* (**Pr. 10**).
3. Then with centre *F*, *and radius FD*, describe the required circle, *which will pass through the given point A, and be tangential to the given line BC, at the given point D.*

Problem 48.

To divide the **area** *of a given* **circle** *A into any number of equal parts by* **concentric circles** (*say three in this case*).

1. Draw *any* radius *AB*, and divide it into *three* equal parts in the points *C*, *D* (**Pr. 15**).

2. On AB describe a semi-circle, and *from the points of division* C and D, erect perpendiculars to AB (**Pr. 2**), meeting the semicircle in E and F.

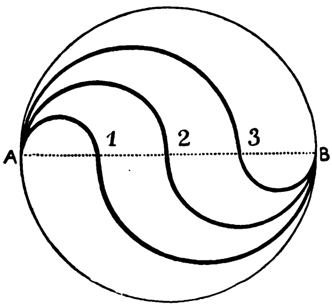


3. From A as centre, *with* AE *and* AF *as radii*, describe circles. *Then the areas* 1, 2, 3, *contained between these circles, will be equal.*

Problem 49.

To divide a given **circle** *into any number of parts, which shall be* **equal** *both in* **area** *and* **outline**.

1. Draw *any* diameter AB, and divide it into the required number of parts (say *four*) (**Pr. 15**) in the points 1, 2, 3.



2. Bisect $A1$, and describe a semicircle *on* it, and a similar one *below* $3B$.

3. Bisect 1B, the remaining part of the line *AB*, and describe a semicircle *below* the line, and *with the same radius* a similar one on the line *A*3.

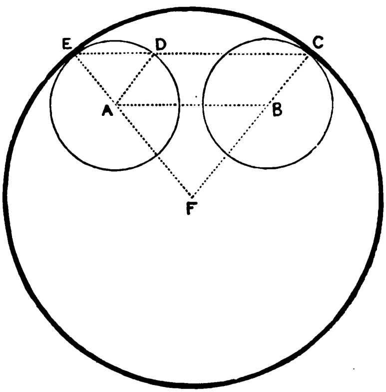
4. With points 1 and 3 as centres, describe semicircles *on* the line *A* 2 and *below* 2 *B* respectively. *Then the given circle will be divided into the required four parts which are equal both in area and outline.*

Problem 50.

To describe a **circle** *touching* **two given circles** *A and B, and one of them in a given* **point** *C.*

1. Join the centres of the two circles *A* and *B* by the straight line *AB*.

2. Draw from *C*, the given point of contact, a radius, *CB*.



3. In the other circle, draw a radius *AD*, parallel to *BC* (**Pr 8**).

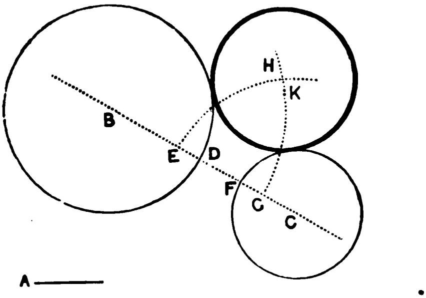
4. Join *CD*, producing it if necessary to a point opposite to *C*, as *E*.

5. Join *CB* and *EA*, and produce them until they meet in *F*. *Then FC or FE will be the radius of the required circumscribing circle.*

Problem 51.

To describe a **circle** *of a given* **radius** *A, touching any* **two given circles** *B and C, tangentially.*

1. Draw a line of *indefinite* length through *B* and *C.*
2. Make *DG* and *FE* each equal to *A.*



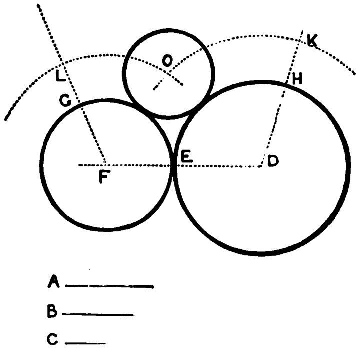
3. With *B* as centre, *and radius BG,* describe the arc *GH.*
4. With *C* as centre, *and radius CE,* describe arc *EK.* With *K* as centre, *and radius A,* describe the required circle, *which shall be tangential to the given circles B and C.*

Problem 52.

To describe **three circles** *having any given* **radii** *A, B, and C, each circle being* **tangential** *to the other two.*

1. Take *any* point *D* as centre, and *with line A as radius,* describe a circle, and produce the radius *DE* outwards *indefinitely.*
2. From *E, with line B as radius,* cut *DE* produced in *F;* and from *F, with the same radius,* describe the second circle.
3. Draw *any other* radius to each circle as *DH, FG,* and produce them.

4. On the produced radii, set off *HK*, and *GL*, equal to *C*.



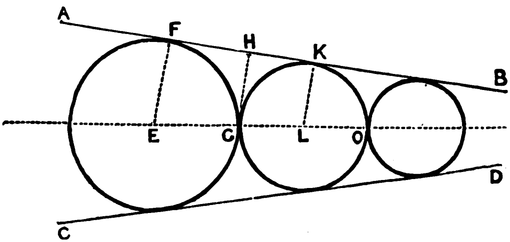
5. From points *D* and *F*, *with DK and FL as radii respectively*, describe arcs cutting each other in *O*.

6. From *O* as centre, *with line C as radius*, describe the remaining circle. *Then F, D, O shall be the three required circles.*

Problem 53.

To describe a **series of circles** *in succession*, **tangential** *to two given* **converging lines** *AB and CD.*

1. Bisect the angle made by the two given converging lines (**Pr.** 12), and take *any* point *E* in the line of bisection.



2. From *E*, draw a perpendicular *EF* to one of the given lines *AB* (**Pr.** 3).

3. *Then E is the centre, and EF the radius of the first circle, which cuts the line of bisection in G.*

4. From *G*, draw a line perpendicular to *EG* (**Pr. 2**), meeting *AB* in *H*.

5. From *H*, set off *HK* equal to *HF*, and draw a line from *K*, perpendicular to *AB*, or parallel to *EF* (**Pr. 8**), meeting the line of bisection in *L*.

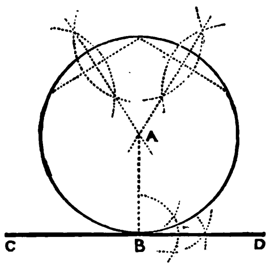
6. *Then L is the centre, and LK or LG the radius of the second circle, which cuts the line of bisection in O.*

NOTE.—By the same construction other circles may be described either towards *A, C*, or *B, D*.

Problem 54.

To draw a **tangent** *to a given* **circle** *A at a given* **point of contact** *B in the* **circumference.**

1. Find the centre of the circle *A* (**Pr. 45**), and from *B* draw a radius *BA*.

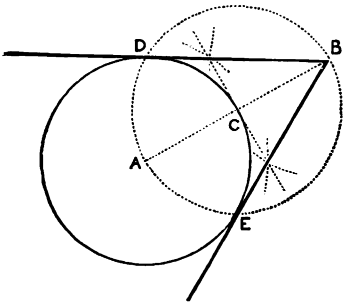


2. From *B* draw a line perpendicular to *AB* (**Pr. 2**), and produce it both ways towards *C* and *D*. *Then CD is the required tangent to the given circle A.*

Problem 55.

To draw a **tangent** *to a given* **circle** *A, from a given* **point** *B* **outside** *the circumference.*

1. Find the centre of the circle *A* (**Pr. 45**), and draw a line from *B* to the centre *A*.



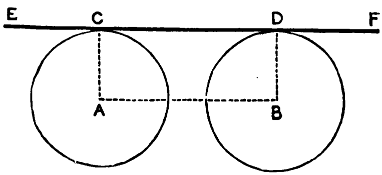
2. Bisect *AB* in the point *C* (**Pr. 1**), and describe the circle of which *AB* is the diameter, and cutting the given circle in the required points of contact *D* and *E*.

3. Join *BD* or *BE*, and *either of these lines produced beyond D or E is the required tangent to the given circle A.*

Problem 56.

To draw a **tangent** *on the* **outside of two given equal circles** *A and B, placed apart.*

1. Join the centres *A* and *B* of the given circles.

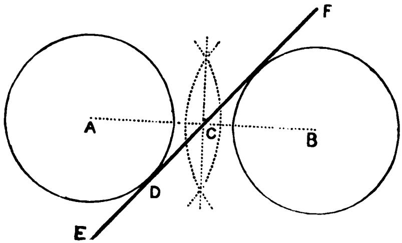


2. At *A* and *B*, draw lines *AC*, *BD* at right angles to *AB*, meeting the circumferences in *C* and *D* (**Pr. 2**), the points of contact.

3. *Draw the required tangent EF through the points C and D.*

Problem 57.

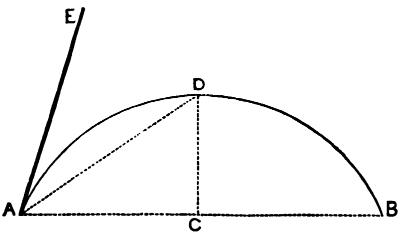
To draw a **tangent between two given equal circles** *A and B, placed apart.*



1. Join the centres *A* and *B* of the given circles.
2. Bisect the line *AB* (**Pr. 1**) in the point *C;* and from *C*, draw a tangent to the circle *A* (**Pr. 55**) one point of contact being at *D*.
3. Produce the line *CD* both ways to *E* and *F*. *Then EF will be the required tangent.*

Problem 58.

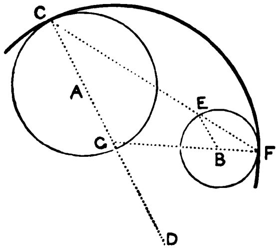
To draw a **tangent** *to any given point of* **contact** *A, in the given* **arc** *of a circle AB, when the* **centre cannot be obtained.**



1. Draw the chord *AB*, and bisect it in *C* (**Pr. 1**).
2. From the point *C*, erect a perpendicular *CD* to *AB* (**Pr. 2**).
3. Join *DA*, and make the angle *DAE* equal to the angle *DAC* (**Pr. 10**).
4. Produce *AE*, *and it is the required tangent.*

Problem 59.

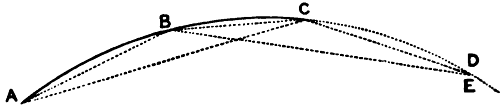
To draw a **tangential arc** *to* **two given circles** *A and B, touching one of the given circles in any given point C.*



1. From C, through centre A, draw CD of *unlimited* length.
2. From centre B, draw BE parallel to CD (**Pr. 8**).
3. From C, through E, draw CF; and from F, through B, draw FG.
4. With G as centre, and *GC as radius,* describe the required tangential arc. *Then the arc CF shall be tangential to the two given circles A and B.*

Problem 60.

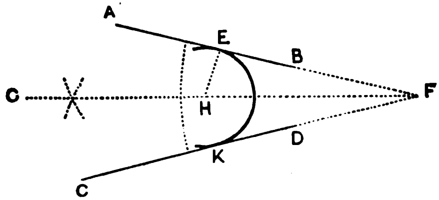
To find a **point** *which would be situated in the* **continuation** *of a given* **arc** *ABC, when the* **centre of the arc cannot be obtained.**



1. Draw *any* two chords AB, BC.
2. Make the angle BCD equal to the angle ABC (**Pr. 10.**)
3. Cut off CE equal to AB; *then point E would be in the continuation of the given arc ABC.*

Problem 61.

To describe the **arc** *of a circle, which shall be* **tangential** *to any two given* **converging lines** *AB, CD, and which shall touch one of the given lines at a given point E.*



1. Produce the converging lines *AB, CD*, until they meet in point *F*, and bisect the angle *AFC* by the line *FG* (**Pr.** 4).
2. From *E*, draw *EH* perpendicular to *AB* (**Pr.** 2).
3. With *H* as centre, *and HE as radius*, describe the required arc. *Then the arc EK shall be tangential to the two given converging lines AB, CD.*

Section V.—POLYGONS.

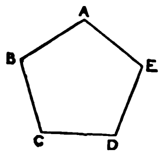
DEFINITIONS.

(In the construction of rectilineal figures, we have hitherto treated of only TRILATERAL *and* QUADRILATERAL *figures. Sections II. and III.)*

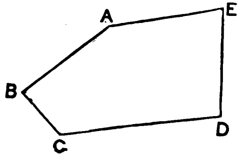
1. "Multilateral figures, or **polygons**, are those which are contained by more than four straight lines" (**Euc. I., Def. 23**).

   NOTE.—A polygon is either *regular* or *irregular*.

2. A **regular polygon** is one that has all its sides and all its angles equal. *Ex. ABCDE*—



3. An **irregular polygon** is one that has its sides and angles unequal. *Ex. ABCDE*—



NOTE.—A polygon may have *any* number of sides, but in *Practical Geometry* we seldom have to deal with figures having more than twelve sides.

4. A **pentagon** is a polygon having **5** sides.

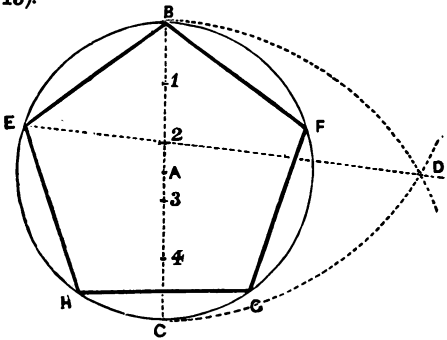
A	**hexagon**	"	"	**6**	"
A	**heptagon**	"	"	**7**	"
An	**octagon**	"	"	**8**	"
A	**nonagon**	"	"	**9**	"
A	**decagon**	"	"	**10**	"
An	**un-decagon**	"	"	**11**	"
A	**do-decagon**	"	"	**12**	"

Problem 62—(A.)

To inscribe **any regular polygon** (*say a* **pentagon**) *in a given* **circle** *A.*

General Method.

1. Draw a diameter *BC*, and divide it into as many *equal* parts *as the polygon is to have sides* (in this case *five*. **Pr. 15**).



2. With points *B* and *C* as centres, *and the diameter BC as radius,* describe arcs cutting at *D.*

3. From *D*, draw a line *through point* 2 to *E.* Join *EB*, *which is one of the sides of the required polygon.*

4. *With EB as radius,* starting from *B*, cut the circle in the points *F*, *G*, *H* successively.

5. Join *BF*, *FG*, *GH*, and *HE* by straight lines, *and a regular pentagon will be inscribed within a given circle A.*

Note.—Whatever number of sides the polygon may have, the line from *D* must always be drawn through the *second* division of the diameter.

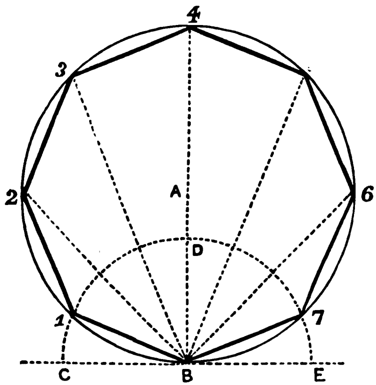
(B.)

To inscribe **any regular polygon** (*say an* **octagon**) *in a given* **circle** *A.*

Another General Method.

1. Draw *any* radius *AB*, and at *B* draw a tangent to the circle (**Pr. 54**).

2. From *B*, *with any radius*, describe a semicircle *CDE*, and divide it into *as many parts as the polygon is to have sides.*



3. Draw lines from *B* through each point of division; produce them, and they will cut the circle in the place of the angles of the polygon.

4. Join the points *B* 1, 1 2, 2 3, &c., and *a regular octagon will be inscribed in the given circle A.*

Problem 63.

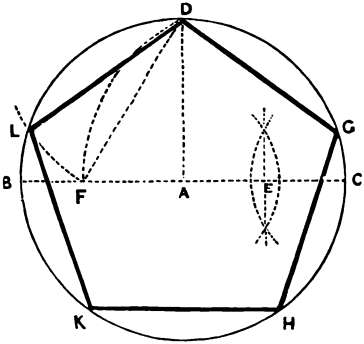
To inscribe a regular **pentagon** *within a given* **circle** *A.*

1. Draw a diameter *BC*, and from the centre *A* erect a perpendicular *AD* (**Pr. 2**).

2. Bisect the radius *AC* in *E* (**Pr. 1**).

3. From E as centre, *with radius ED*, describe an arc DF, cutting the diameter in F.

4. Draw the chord of the arc DF. This will be the *length* of a side of the pentagon.

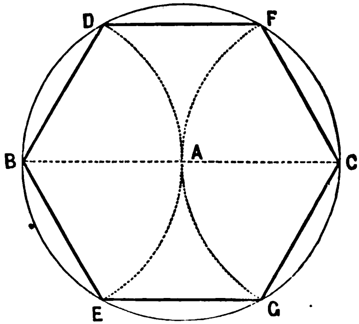


5. *With DF as radius* starting from D, cut the circle in the points G, H, K, and L successively.

6. Join DG, GH, &c., by straight lines, and *a regular pentagon will be inscribed within a given circle A.*

Problem 64.

To inscribe a regular **hexagon** *within a given* **circle** *A.*



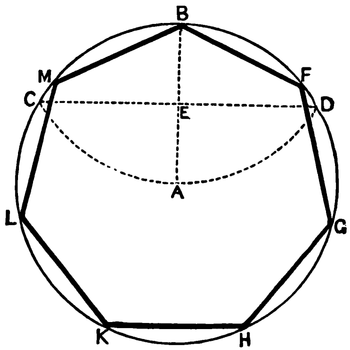
1. Draw *any* diameter BC.

2. With *B* and *C* as centres, *and the radius of the circle AB*, describe the arcs *DAE* and *FAG*.
3. Join *BD*, *DF*, &c., and *the required regular hexagon is inscribed in the given circle A.*

Problem 65.

To inscribe a regular **heptagon** *within a given* **circle** *A.*

1. Draw *any* radius *AB*, and from *B*, *with BA as radius*, describe an arc *CAD* cutting the circumference in points *C* and *D*.



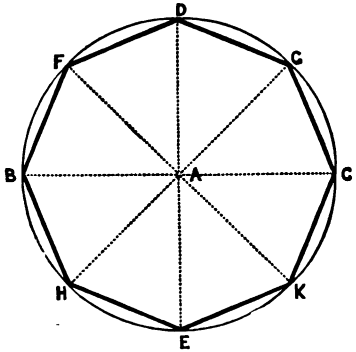
2. Join *CD* by a straight line cutting *AB* in *E*. Then *EC* or *ED* will be the *length* of a side of the heptagon.
3. *With EC or ED as radius*, starting from *B*, cut the circle in the points *F*, *G*, . . . *M* successively.
4. Join *BF*, *FG*, &c., by straight lines, and *a regular heptagon will be inscribed within a given circle A.*

Problem 66.

To inscribe a regular **octagon** *within a given* **circle** *A.*

1. Draw *any* diameter *BC*, and bisect it by another diameter *DE*.

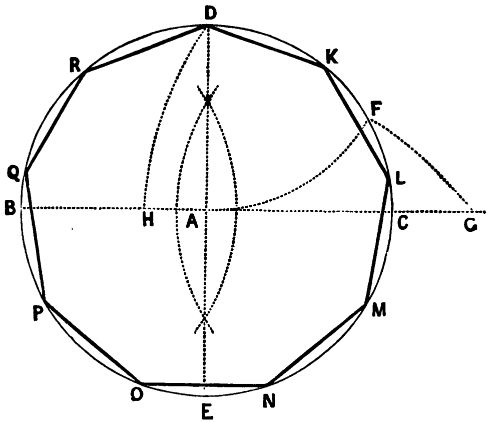
2. Bisect each of the four arcs by the diameters *FK*, *HG*.



3. Join the points *BF*, *FD*, &c., by straight lines, and *the required regular octagon is inscribed in the given circle A.*

Problem 67.

To inscribe a regular **nonagon** *within a given* **circle** *A.*



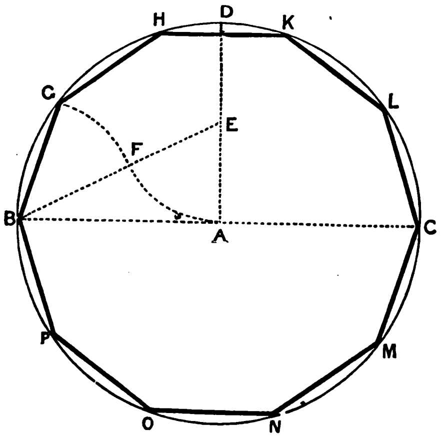
1. Draw a diameter *BC*, and produce it one way *indefinitely* (say to the *right*), and bisect *BC* by another diameter *DE*.

2. From *D*, *with radius DA*, describe an arc cutting the arc *DC* in *F*.
3. From *E*, *with radius EF*, cut the produced diameter *BC* in *G*.
4. From *G*, *with radius GD*, cut the diameter *BC* in *H*.
5. With *BH*, *which is equal to a side of the nonagon*, cut the circle, starting from *D*, in the points *K*, *L* ...... *R*. Join these points by straight lines, *and the required regular nonagon is inscribed in the given circle A.*

Problem 68.

To inscribe a regular **decagon** *within a given* **circle** *A*.

1. Draw *any* diameter *BC*, and a radius *AD* perpendicular to it from the centre of the circle (**Pr. 2**).



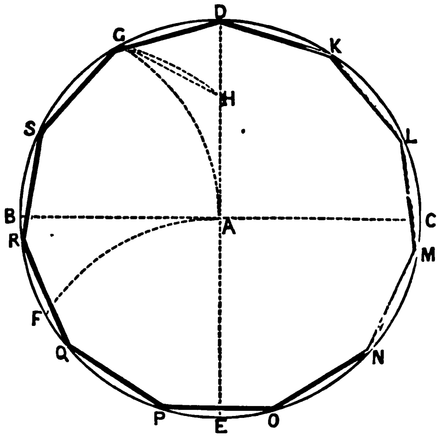
2. Bisect *AD* in *E* (**Pr. 1**), and join *BE*.
3. From *E*, *with EA as radius*, describe an arc cutting *BE* in *F*.
4. From *B*, *with BF as radius*, describe an arc cutting the circumference in *G*.
5. Draw the straight line *BG*. *It will be a side of the decagon.*

6. *With the straight line BG as radius*, starting from *G*, cut the circle in the points *H*, *K* . . . *P* successively.
7. Join *GH*, *HK*, &c., by straight lines, and *a regular decagon will be inscribed within a given circle A.*

Problem 69.

To inscribe a regular **un-decagon** *within a given* **circle** *A.*

1. Draw two diameters *BC* and *DE* perpendicular to each other, and cutting each other in *A*.

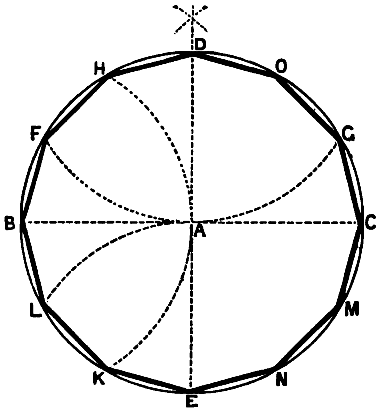


2. From *E*, *with radius EA*, describe an arc, cutting the quadrant *EB* in *F*.
3. From *B*, *with the same radius*, describe an arc, cutting the quadrant *BD* in *G*.
4. From *F*, *with radius FG*, describe an arc, cutting the radius *AD* in *H*.
5. Draw the straight line *GH*, *it will be equal to a side of the un-decagon.*
6. *With the straight line GH as radius*, starting from *D*, cut the circle in the points *K*, *L*, . . . *G* successively.
7. Join *DK*, *KL*, &c., by straight lines, and *a regular un-decagon will be inscribed within a given circle A.*

Problem 70.

To inscribe a regular **do-decagon** *within a given* **circle** *A.*

1. Draw *any* two diameters *BC* and *DE* at right angles to each other.



2. With centres *D, B, E,* and *C,* and the radius of the circle *AB,* describe arcs cutting the circumference in *F, G, H, K, L, M, N,* and *O.*

3. Join *DO, OG,* &c., by straight lines, and *a regular do-decagon will be inscribed within a given circle A.*

Problem 71—(A.)

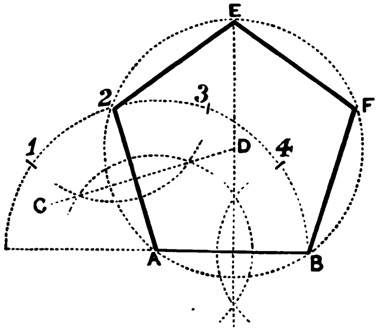
To construct **any regular polygon** (*say a* **pentagon**) *on a* **given straight line** *AB.*

General Method.

1. Produce the side *AB,* in this case, say towards the left. With *A* as centre *and AB as radius,* describe the semicircle.

2. Divide the semicircle into as many *equal* parts as the polygon is to have sides (five).

3. From A draw A 2, to the *second* division of the semicircle. *This makes another side of the required figure.*



4. Bisect the two sides 2 A, AB, by lines CD, DE, and from centre D, their point of intersection, *and radius* DA, describe the circumscribing circle.

5. Mark off, on the circumference, the divisions 2 E, EF, *equal* to AB. Join 2 E, EF, FB, *and the pentagon is constructed on the given line* AB.

NOTE.—A line must always be drawn from A to the *second* division on the semicircle, no matter how many sides the polygon is to have.

(B.)

Another General Method.

1. At point B raise a perpendicular BC equal to AB (**Pr. 2**), and describe the quadrant AC.

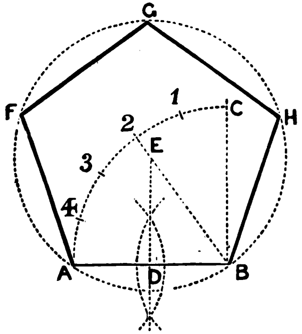
2. Divide AC into as many *equal* parts as the required polygon is to have sides (five).

3. Draw a line from B to the *second* point of division.

4. Bisect AB in D (**Pr. 1**), and from D erect a perpendicular to meet B 2 in E. (**Pr. 2**).

5. From centre E, *with* EA *radius*, describe a circle; it will contain the required polygon.

6. *With AB as radius*, starting from *A*, cut the circle in the points *F*, *G*, *H* successively.

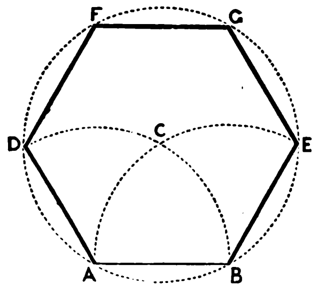


7. Join *AF*, *FG*, &c., by straight lines, and *a regular pentagon will be constructed upon a given straight line AB.*

Problem 72.

To construct a regular **hexagon** *on a given* **line** *AB.*

1. With points *A* and *B* as centres, *and radius AB*, describe the arcs intersecting at *C*.



2. From the point *C*, *with CA as radius*, describe the circle.
3. From *D* and *E*, *with the same radius*, cut off *F* and *G*.
4. Join *AD*, *DF*, &c., by straight lines, *and ADFGEB is the required hexagon.*

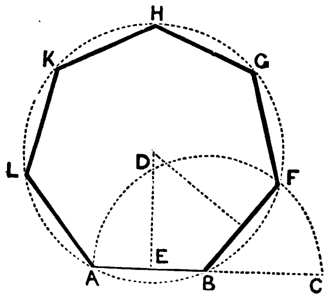
Problem 73.

To construct a regular **heptagon** *on a given* **line** *AB.*

1. From *B* as centre, *with radius AB,* describe a semicircle cutting *AB* produced in *C.*

2. From *A*, *with the same radius,* cut the semicircle in *D.*

3. Bisect *AB* in *E* (**Pr. 1**), and join *DE.*



4. From *C*, *with DE as radius,* cut the semicircle in *F.*

5. Join *BF ; it is another side of the heptagon.*

6. Find the centre of the circle that contains it, and complete the heptagon. *Then AB......L will be the heptagon required.*

Problem 74.

To construct a regular **octagon** *on a given* **line** *AB.*

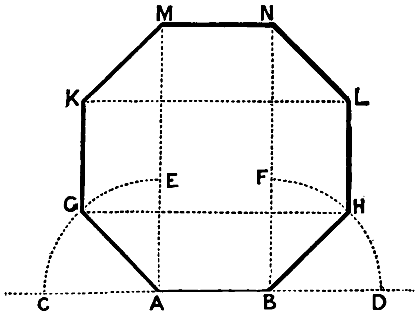
1. Produce *AB* both ways, and erect perpendiculars at *A* and *B* (**Pr. 2**).

2. From *A* and *B*, *with radius AB,* describe the quadrants *CE, FD.*

3. Bisect these quadrants in the points *G* and *H* respectively.

4. Join *AG, BH ; these will be two more sides of the octagon.*

5. Join *GH*, and at *G* and *H* erect perpendiculars *GK*, *HL*, equal to *AB* (**Pr. 2**).



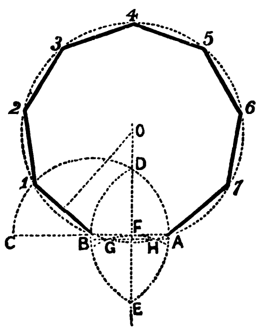
6. Join *KL*, and make the perpendiculars at *A* and *B* equal to *GH* or *KL*—viz., *AM* and *BN*.

7. Join *KM*, *MN*, and *NL*, and *the required octagon will be constructed on the given line AB.*

Problem 75.

To construct a regular **nonagon** *on a given* **line** *AB.*

1. Produce the line *AB*; and from *B*, *with radius BA*, describe an arc cutting the produced line *AB* in *C*, and being produced below *A*.



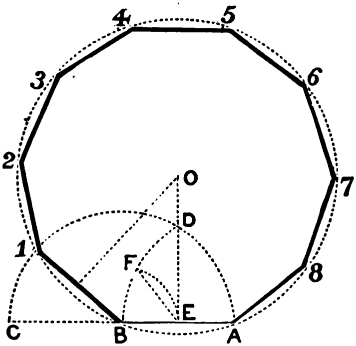
2. From *A*, *with the same radius*, describe an arc, cutting the first arc in *D* and *E*.

3. Draw line *DE*, cutting *AB* in *F*.
4. From *D*, *with radius DA*, describe arc *AB*.
5. From *E*, *with radius EF*, describe an arc, cutting the arc *AB* in *G* and *H*.
6. From *C*, *with line GH as radius*, cut the semicircle in **1.**
7. Draw line *B* 1 ; *it is a second side of the nonagon.*
8. Bisect *B* 1, and obtain *O*, the centre of the circle.
9. Mark off, on the circumference, the divisions 1 2, 2 3, &c., *equal* to *B* 1. Join 1 2, 2 3, &c., *and a nonagon is constructed on the given line AB.*

Problem 76.

To construct a regular **decagon** *on a given* **line** *AB.*

1. Produce the line *AB*, and from *B*, *with radius BA*, describe a semicircle, cutting it in *C*.



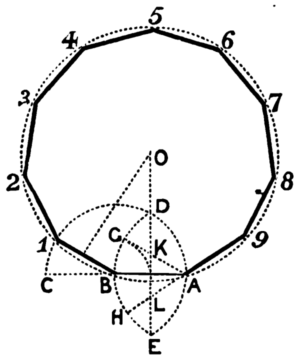
2. From *A*, *with radius AB*, describe an arc, cutting the semicircle in *D*, and bisect *AB* in *E* (**Pr. 1**).
3. From *B*, *with radius BE*, describe an arc, cutting arc *BD* in *F*.
4. Draw line *EF*.
5. From *C*, *with radius EF*, cut the semicircle in 1 ; then *B* 1 *is a second side of the decagon.*

6. Bisect *B* 1, and obtain *O*, the centre of the circle.
7. Mark off, on the circumference, the divisions 1 2, 2 3, &c , *equal* to *B* 1. Join 1 2, 2 3, &c., *and a decagon is constructed on the given line AB.*

Problem 77.

To construct a regular **un-decagon** *on a given* **line** *AB.*

1. Produce the line *AB*, and from *B*, *with radius BA*, describe an arc, cutting the produced line *AB* in *C*, and being produced below *A*.
2. From *A*, *with the same radius*, describe an arc, cutting the first arc in *D* and *E*.
3. Draw line *DE*, bisecting *AB*.

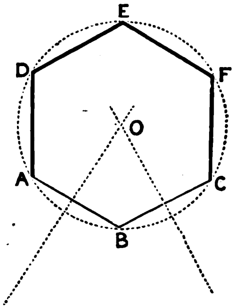


4. From *B*, *with half BA as radius*, describe an arc cutting *BD* in *G*.
5. Bisect the arc *BE* in *H;* and draw *AG*, *AH*, cutting *ED* in *K* and *L*.
6. From *C*, *with radius AL*, cut off *C* 1 on the semicircle.
7. Draw line *B* 1 ; *it is a second side of the un-decagon.*
8. Bisect *B* 1, and obtain *O*, the centre of the circle.
9. Mark off, on the circumference, the divisions 1 2, 2 3, &c., equal to *B* 1. Join 1 2, 2 3, &c., and *an un-decagon is constructed on the given line AB.*

Problem 78.

To **complete** *a regular* **polygon,** *its* **two sides** *AB, BC being given.*

1. Bisect the lines *AB, BC* by perpendiculars meeting at *O* (**Pr. 1**).

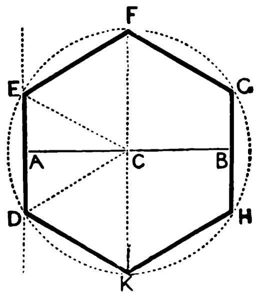


2. With centre *O, and radius OA,* describe the circle.
3. From *A*, mark off the distance *AB* to *DE,* &c. Join *AD, DE, EF,* &c., and *a regular polygon will be completed*—in this case a **hexagon.**

Problem 79.

To construct a regular **hexagon,** *its* **diameter** *AB being given.*

1. Bisect *AB* in *C* (**Pr. 1**).



2. Through the point *A* draw a line *DE* perpendicular to *AB* (**Pr. 2**).

3. On *CA*, as an altitude, construct an equilateral triangle, having its vertical angle at *C* (**Pr. 19**).

4. From *C*, *with radius CE or CD*, describe a circle.

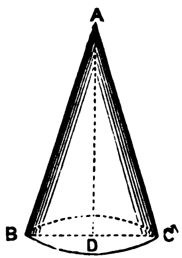
5. From point *E*, mark off the distance *ED* to *FG*, &c. Join *EF*, *FG*, &c., and *a regular hexagon will be constructed, having the given diameter AB*.

Section VI.—ELLIPSES, &c.

DEFINITIONS.

In order to understand the following definitions **clearly,** *we must refer to that* SOLID *which is called a* CONE.

1. A **cone** is a solid figure, the base of which is a circle, but which tapers to a point from the base upward. *Ex. ABC*—



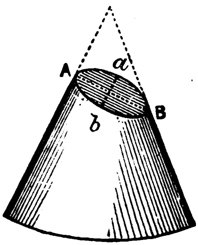
NOTE 1.—A straight line drawn from the centre of the base to the *apex* (or summit) is called its *axis*. *Ex. AD*—

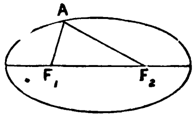
NOTE 2.—When the apex is *perpendicular* to the base, the cone is said to be a *right* cone.

NOTE 3.—When the axis is *not* perpendicular to the base, the cone is said to be an *oblique* cone.

NOTE 4.—If a right cone be cut in two parts by a plane parallel to the base, the section will be *similar* to the base, *i.e.*, a circle. But if a cone be cut in some *other* way, the section has a distinctive name. Thus—

2. An **ellipse** is a section of a cone, produced by a plane which is not parallel to the base. *Ex. AB*—

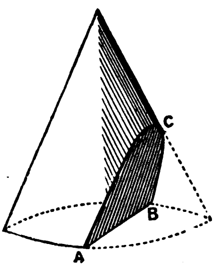


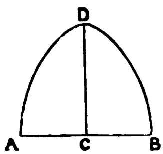


NOTE 1.—Such a figure has *two* diameters, *unequal* in length; viz., the *long* diameter *AB*, called the *transverse* or major **axis**, and the *short* diameter *ab*, called the *conjugate* or minor axis.

NOTE 2.—There are two important fixed points in the transverse axis called *foci* (singular, *focus*) equally distant from the centre, and are such that the sum of two straight lines F_1A, AF_2 drawn from them to *any* point *A* in the circumference, is equal to the length of the major axis.

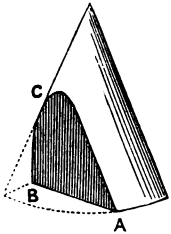
3. A **parabola** is a section of a cone, produced by a plane which is parallel to one of the sides. *Ex. ABC*—

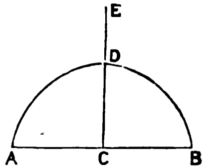




NOTE.—Its base *AB* is termed its *double ordinate*, *AC* or *CB* being its *ordinate*; and its altitude *CD* is called its ***abscissa***.

4. A **hyperbola** is a section of a cone, produced by a plane which is parallel to its axis. *Ex. ABC*—

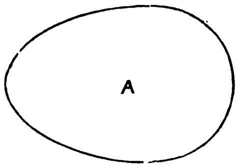




NOTE 1.—*AB* is termed its *double ordinate*, *AC* its *ordinate*, *CD* its *abscissa*, and *CE* its *diameter*.

NOTE 2.—The three foregoing sections are usually known as the "**conic sections.**"

5. An **oval**, as its name implies, is simply an *egg*-shaped figure, being wider at one end than at the other. *Ex. A*—

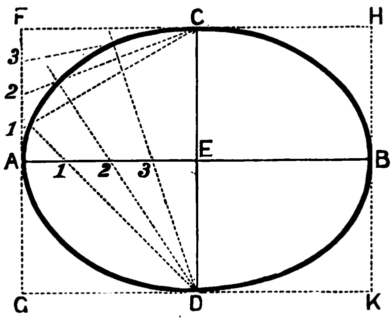


Problem 80.

To describe an **ellipse,** *its* **axes** *or* **transverse** *and* **conjugate diameters** *AB and CD being given.*

1. Place the transverse diameter *AB* and the conjugate diameter *CD* perpendicular to each other at their centres *E*.
2. Through *A* and *B* draw the lines *FG*, *HK* parallel to *CD* (**Pr. 8**), and through *C* and *D* draw *FH*, *GK* parallel to *AB*, forming the rectangle *GFHK*.
3. Divide *AE* and *AF* into *any* number of equal parts, in this case *four* (**Pr. 15**).

Draw lines 1 *C*, 2 *C*, 3 *C*; and from *D*, through points 1, 2, 3 in the transverse diameter, draw lines which will intersect the former lines. *The points of intersection will be in the curve of the ellipse required.*



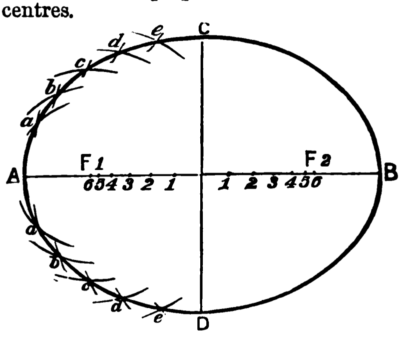
Note 1.—By repeating the process in the other divisions of the rectangle, the curve of the required ellipse will be completed.

Note 2.—The ellipse must be carefully drawn by *hand.*

N.B.—*This method is that of intersecting lines.*

Another Method.

1. Place the transverse diameter *AB* and the conjugate diameter *CD* perpendicular to each other at their centres.



2. From *C* or *D*, *with half AB as radius,* describe arcs, cutting *AB* in *F* 1, *F* 2. *These points are the foci of the ellipse.*

3. From *F* 1 to the centre of *AB*, mark off *any* number of parts, as 1, 2, 3, 4, &c., and it will be more convenient if the divisions *lessen* as they approach *F* 1.

4. From *F* 1, *with radius A* 1, *A* 2, *A* 3, &c., describe arcs in the spaces *AC* and *AD*.

5. From *F* 2, *with B* 1 (the first division towards *A* beyond the centre of *AB*), *B* 2, *B* 3, *&c., as radius*, describe arcs cutting the arcs already described from *F* 1; radius *B* 1 cutting arc *A* 1, &c., in *a*, *b*, *c*, *d*, &c. *The points of intersection will be in the curve of the ellipse required.*

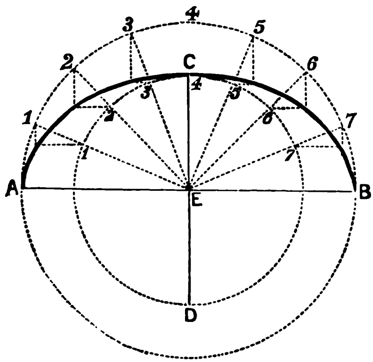
NOTE 1.—By repeating the process in the spaces *BC*, *BD*, the curve of the required ellipse will be completed.

NOTE 2.—The ellipse must be carefully drawn by *hand.*

N.B.—*This method is that of intersecting arcs.*

Another Method.

1. Place the diameters perpendicular to each other at their centres *E*, as before.



2. From *E*, *with radii EC and EA*, describe circles.

3. Divide the circumference of the larger circle into *any* number of equal parts, 1, 2, 3, 4, &c.

4. Draw radii from each point of division, cutting the circumference of the smaller circle also in 1, 2, 3, 4, &c.

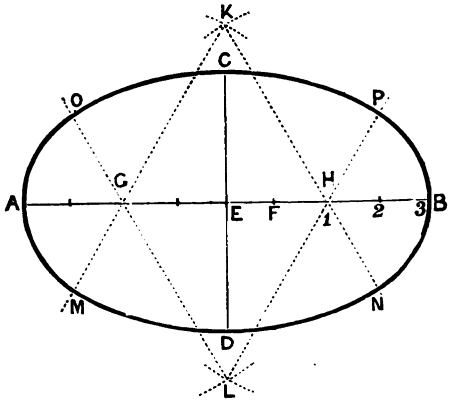
5. From the divisions of the smaller circle, draw lines parallel to the transverse axis *AB*.

6. From the divisions of the larger circle, draw lines parallel to the conjugate axis *CD*. *The points of intersection will be in the curve of the ellipse required.*

Another Method.

1. Place the given diameters *AB*, *CD* perpendicular to each other at their centres *E*.

2. From *A*, *with CD as radius*, mark the point *F*.



3. Divide *FB* into *three* equal parts.

4. From *E*, *with two of these parts as radius*, cut *AB* in *G* and *H*.

5. From *G* and *H*, *with GH as radius*, describe arcs in *K* and *L*.

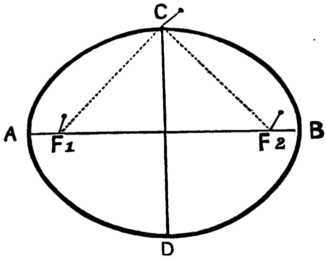
6. From *K* and *L*, *with radius KD*, describe arcs *MN*, *OP*; and from *G* and *H*, *with radius HB*, describe arcs *MO*, *PN*, *which complete the required ellipse.*

Note.—Lines drawn from *K* and *L*, through *G* and *H*, will show where the four arcs unite.

N.B.—*This method is by means of arcs of circles.*

Another Method.

1. Place the given diameters, as before, and find the foci *F* 1, *F* 2 (**Pr. 80**).
2. Fix a pin at each of the foci, and another at one end of the conjugate diameter, as *C*.



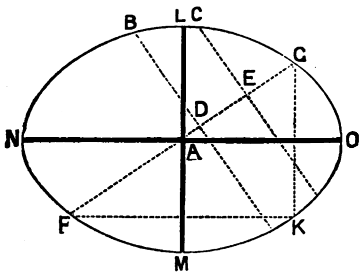
3. Tie a piece of thread tightly around the three pins, forming a triangle *C*, *F* 1, *F* 2.
4. Take out the pin at *C*, and put a pencil within and against the string at *C*, and keeping the string perfectly tight, and close to the paper throughout, *describe the curve of the required ellipse, which will pass through B, D, A, C.*

N.B.—*This method is by mechanical means.*

Problem 81.

To find the **centre** *and* **axes** *of a given* **ellipse** *A.*

1. Draw *any* two chords *B* and *C* parallel to each other (**Pr. 8**), and bisect them in *D* and *E* (**Pr. 1**).



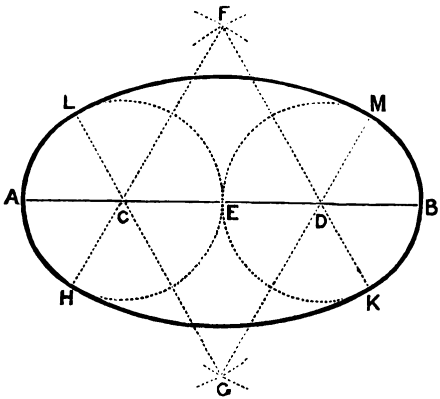
2. Draw a diameter *FG* through *D* and *E*, and bisect it in *A* ; then *A is the centre of the ellipse.*

3. From A, *with AG as radius*, mark the point K, and join GK and KF.
4. Through A draw LM and NO parallel to GK and FK (**Pr. 8**); then *NO and LM are the axes required.*

Problem 82.

To describe an **elliptical figure,** *one* **diameter** *AB being given.*

1. Divide AB into four equal parts, in points C, E, D (**Pr. 15**).
2. From C and D, *with radius CA or DB*, describe circles touching each other in E.



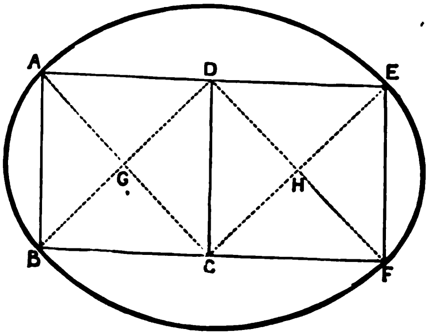
3. From C and D, *with radius CD*, describe arcs cutting each other in F and G.
4. Draw lines FC, FD, GC, GD, and produce them until they cut the circles in H, K, L, M.
5. From F and G, *with radius FH or GM*, draw arcs uniting H with K and L with M, *which will complete the required elliptical figure.*

Problem 83.

To construct an **elliptical figure, two squares** *$ABCD$ and $CDEF$ being given.*

1. Draw diagonals in each of the squares, intersecting each other in G and H.

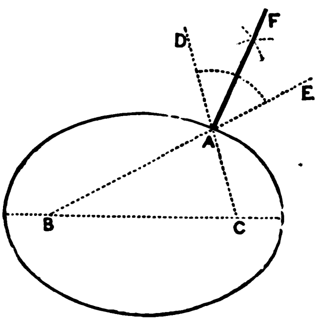
2. From *C, with radius CA or CE,* **describe the arc** ***AE.***
3. From *D, with the same radius,* **describe the arc** ***BF.***



4. From *G, with radius GA,* describe the arc *BA.*
5. From *H, with the same radius,* describe the arc *EF, which will complete the required elliptical figure.*

Problem 84

To draw a **perpendicular** *to the curve of a given* **ellipse** *from a given* **point** ***A.***

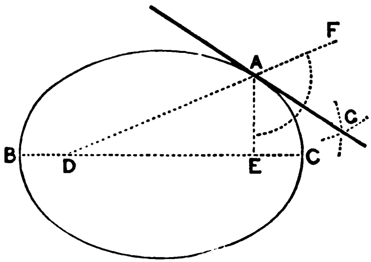


1. Draw the transverse axis, and find the foci *B* and *C* (**Pr. 80**).
2. Draw the lines *BA* and *CA*, and produce them, making the angle *DAE*.
3. Bisect angle *DAE* by line *AF* (**Pr. 4**); then *AF is perpendicular to the curve of the given ellipse.*

F

Problem 85.

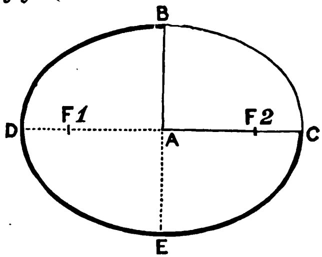
To draw a **tangent** *to the curve of a given* **ellipse** *at a given* **point of contact** *A.*



1. Draw the transverse axis *BC* (**Pr. 81**), and obtain the foci *D* and *E* (**Pr. 80**).
2. From *D* and *E* draw lines *DA*, *EA* through the given point of contact *A*, producing one of them, as *DA* to *F*.
3. Bisect the external angle *FAE* in *G* (**Pr. 4**).
4. Draw line *GA*, and produce it; then *GA is a tangent to the given ellipse, through the given point of contact A.*

Problem 86.

To **complete** *the curve of an* **ellipse** *which is partly constructed, one* **quarter** *ABC being given.*

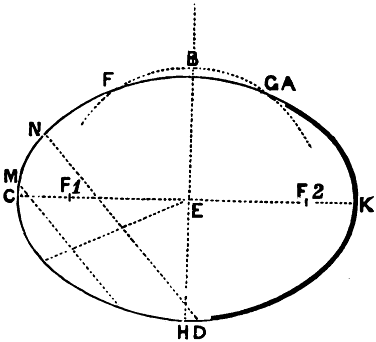


1. Produce *CA* beyond *A*, and make *AD* equal to *AC*; also produce *BA* beyond *A*, and make *AE* equal to *AB*.
2. Find the foci *F* 1 and *F* 2 (**Pr. 80**), and proceed as in **Problem 80.** *The curve BDEC is the required completion.*

Problem 87.

To **complete** *the curve of an* **ellipse** *which is partly constructed,* **more than half** *of the curve ABCD being given.*

1. At some portion of the given curve, not opposite the part which is incompleted, draw two parallel chords *M*, *N*, and find the centre *E* (**Pr. 81**).



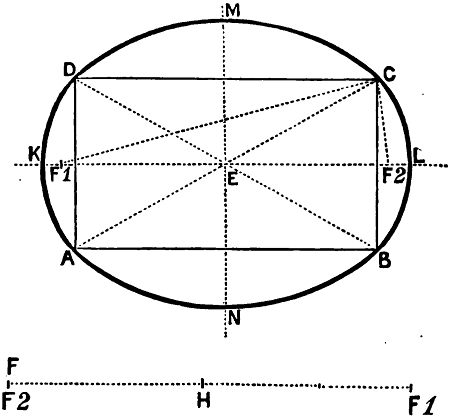
2. From *E*, *with any sufficient radius*, describe an arc, cutting the curve in *F* and *G*.
3. Bisect the arc *FG* in *B* (**Pr. 1**), and produce the line of bisection through *E* to *H*; then *BH is the conjugate axis.*
4. At *E*, draw a line *EC* at right angles to *BH* (**Pr. 2**). meeting the given curve in *C*.
5. Produce *CE*, and make *EK* equal to *EC*; then *CK is the transverse axis.*
6. Find the foci *F* 1, *F* 2 (**Pr. 80**), and complete the curve by means of intersecting arcs (**Pr. 80**). *The curve drawn from A to D through K is the required completion.*

Problem 88.

To describe an **ellipse** *about a given* **rectangle** *ABCD.*

1. Draw the diagonals *AC*, *BD*, meeting in *E*, the centre of the required ellipse, and draw the diameters indefinitely beyond the sides of the given rectangle.

2. From *E*, *with any radius*, cut the produced long diameter both ways in *F* 1, *F* 2, the foci, and join them to one angle of the rectangle, as *C*.



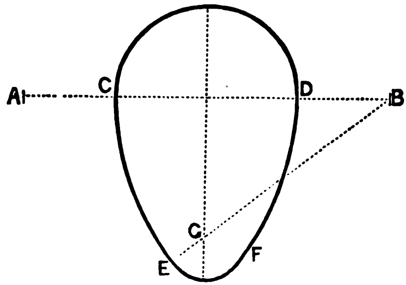
3. Draw *any* straight line, as *FG*, equal to the sum of *CF* 1, *CF* 2, and bisect it in *H* (**Pr. 1**).
4. Make *EK* and *EL* equal to *HF* or *HG;* then *KL is the transverse axis.*
5. From *F* 1, *with half the transverse axis, as EK, as radius*, cut the line perpendicular to *KL* in *M* and *N;* then *MN is the conjugate axis.*
6. Complete the required ellipse by **Problem 80.**

Problem 89.

To describe an **oval** *by arcs of circles.*

1. Draw any straight line *AB*, and describe a semicircle *CD* equal in diameter to the proposed oval.
2. From *C* and *D*, *with the radius of the semicircle*, cut the straight line in *A* and *B*.
3. From *A* and *B*, *with radius BC*, describe the arcs *DF* and *CE*.

4. From A or B, draw a straight line through the transverse diameter, cutting it in G, and meeting the opposite arc in E or F.

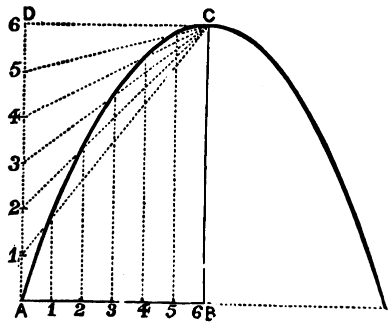


5. From G, *with radius* GE, describe arc EF, *which will complete the required oval* $CDFE$.

NOTE.—The oval may be made longer or shorter by increasing or diminishing the transverse diameter.

Problem 90.

To construct a **parabola,** *its* **ordinate** AB *and* **abscissa** BC *being given.*



1. Through C draw a line CD, parallel and equal to AB (**Pr. 8**), and join AD.

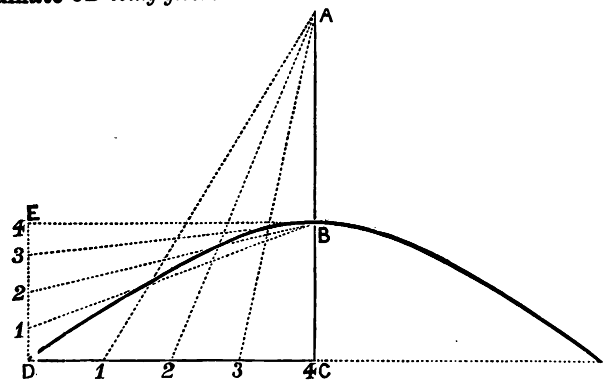
2. Divide AB and AD into the *same* number of equal parts (say *six*).

3. From *C*, draw lines to the points of division in *AD*.
4. From the points of division in *AB*, draw lines parallel to *BC*, till each meets the corresponding line from *AD*. *The points of intersection will be in the curve of the required parabola.*

NOTE.—By repeating the process in the other half, the curve of the required parabola will be completed.

Problem 91.

To construct a **hyperbola,** *its* **diameter** *AB,* **abscissa** *BC, and* **ordinate** *CD being given.*



1. Through *B* draw a line *BE*, parallel and equal to *CD* (**Pr.** 8), and join *DE*.
2. Divide *DC* and *DE* into the *same* number of equal parts (say *four*).
3. From *B*, draw lines to the points of division in *DE*.
4. From *A*, draw lines to the points of division in *DC*. *The points of intersection will be in the curve of the required hyperbola.*

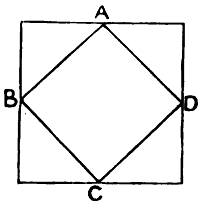
NOTE.—By repeating the process in the other half, the curve of the required hyperbola will be completed.

Section VII.—INSCRIBED FIGURES.

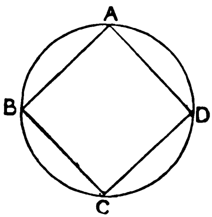
DEFINITIONS.

1. **Inscribed figures.** Inscribed figures are either *rectilineal* or *circular*.

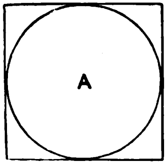
(*a.*) A rectilineal figure is said to be *inscribed in* another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed. *Ex. ABCD*—



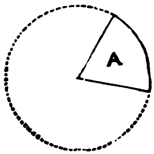
(*b.*) A rectilineal figure is said to be *inscribed in* a circle, when all the angles of the inscribed figure are upon the circumference of the circle. *Ex. ABCD*—



Note.—A *circle* is said to be *inscribed in* a rectilineal figure, when the circumference of the circle touches each side of the figure. *Ex. A*—

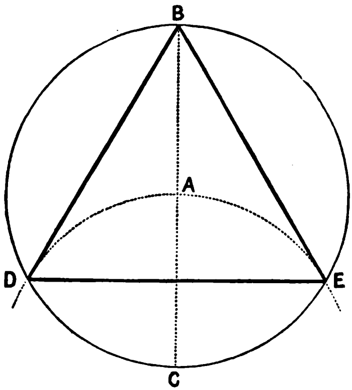


2. A **sector** of a circle is a figure contained by two radii and the intercepted arc. *Ex. A*—



Problem 92.

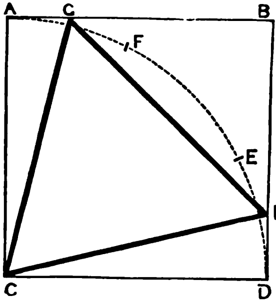
To inscribe an **equilateral triangle** *within a given* **circle** *A*.



1. Find the centre of the circle *A* (**Pr. 45**), and draw a diameter *BC*.
2. From *C* as centre, *with radius CA*, describe an arc cutting the circumference in *D* and *E*.
3. Join *DE*, *EB*, *BD*; then *DEB is the required equilateral triangle inscribed within the given circle A*.

Problem 93.

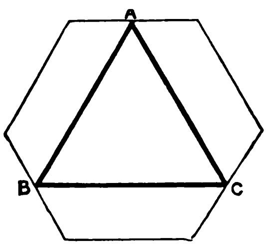
To inscribe an **equilateral triangle** *in a given* **square** *ABCD.*



1. From *C*, *with radius AC*, describe the quadrant *AD*.
2. From *A* and *D*, *with the same radius*, cut off *AE* and *DF*.
3. Bisect *AF* and *ED*, and through the points of bisection draw the lines *CG*, *CH*, cutting the sides of the square in *G*, *H*.
4. Draw *GH*; then *GCH is the required equilateral triangle, inscribed in the given square ABCD.*

Problem 94.

To inscribe an **equilateral triangle** *in a given* **hexagon,** *so that its sides are parallel to three sides of the hexagon.*



1. Bisect the *alternate* sides of the given hexagon (**Pr. 1**) in the points *A*, *B*, and *C*.

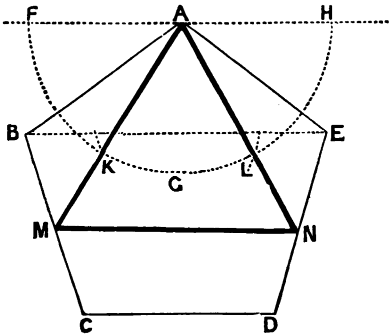
2. Join these points, and *an equilateral triangle will be inscribed in the given hexagon.*

NOTE.—By joining the three alternate *angles* of the hexagon, the largest equilateral triangle it will contain will be inscribed.

Problem 95.

To inscribe an **equilateral triangle** *in a given regular* **pentagon** *ABCDE.*

1. From *A* as centre, *with any radius,* describe a semicircle *FGH.*



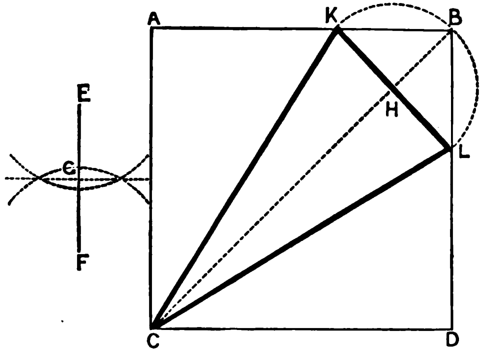
2. From *F* and *H*, *with the same radius,* describe arcs cutting the semicircle in *K* and *L.*
3. From *A*, draw lines through *K* and *L*, meeting the sides of the pentagon in *M* and *N* respectively.
4. Join *MN*, and *AMN will be the required equilateral triangle, inscribed in the given pentagon ABCDE.*

Problem 96.

To inscribe an **isosceles triangle** *within a given* **square** *ABCD, having a given* **base** *EF.*

1. Draw a diagonal *BC*, and bisect *EF* in *G* **(Pr. 1)**.
2. From *B*, mark off, on the diagonal *BC*, *BH* equal to *EG* or *GF.*

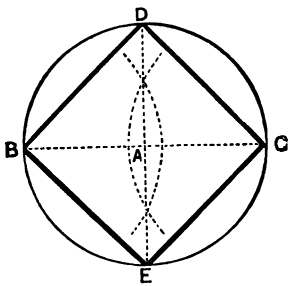
3. With H as centre, *and HB radius*, cut the sides of the square AB and BD in the points K and L.



4. Join CK, KL, and LC, and *an isosceles triangle CKL will be inscribed within the given square ABCD.*

Problem 97.

To inscribe a **square** *within a given* **circle** *A.*

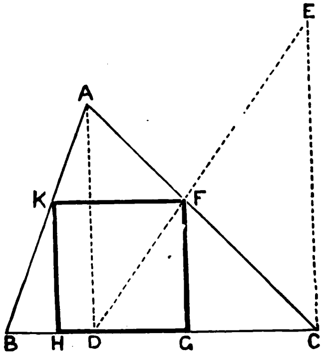


1. Find the centre of the circle A (**Pr. 45**).
2. Draw a diameter BC, and bisect it by another diameter DE.
3. Join BD, DC, CE, and EB; then *BDCE is the square inscribed within the given circle A.*

Problem 98.

To inscribe a **square** *within a given* **triangle** *ABC.*

1. Draw *AD*, the altitude of the given triangle (**Pr. 21**).
2. At point *C* raise a perpendicular *CE* (**Pr. 2**), and make it equal to the base *BC*.



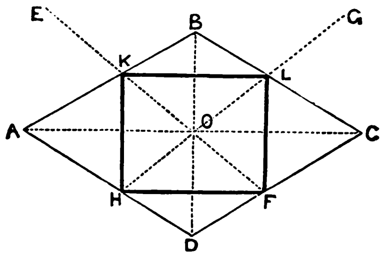
3. Draw the line *ED*, cutting *AC* in *F*.
4. From *F*, let fall a perpendicular *FG* on the base *BC* (**Pr. 3**); then *FG is one side of the required square.*
5. From *G*, mark off the length *FG* on the base *BC* in *H*; and from *H*, with the same length, cut *AB* in *K*.
6. Join *HK*, *KF*; then *KFGH is a square inscribed in the given triangle ABC.*

Problem 99.

To inscribe a **square** *within a given* **rhombus** *ABCD.*

1. Draw the two diagonals *AC*, *BD*.
2. Bisect the two angles *AOB*, *COB* (**Pr. 4**) by the lines *EF*, *GH*, cutting the sides of the rhombus in *K* and *L*.

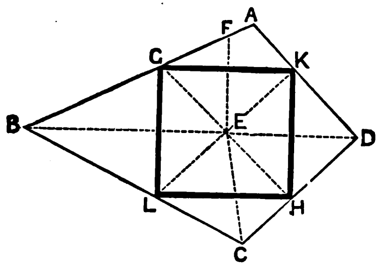
3. Join *FH*, *HK*, *KL*, and *LF*; then *FHKL is the required square, inscribed within the given rhombus ABCD.*



Problem 100.

To inscribe a **square** *in a given* **trapezium** *ABCD, which has its* **adjacent pairs of sides equal.**

1. Draw a diagonal *BD*, bisecting the trapezium and the angle at *B*.



2. Find the centre of the figure in point *E* by bisecting another angle, as at *C* (**Pr. 4**).
3. At point *E* raise a perpendicular to *BD*, as *EF*.
4. Bisect the right angles on either side of *EF*, and produce the lines of bisection to cut the trapezium in *GHKL*.
5. Join *GK*, *KH*, *HL*, and *LG*, and *the figure GKHL will be the required square, inscribed in the given trapezium ABCD.*

Problem 101.

pentagon *ABCDE.*

igles to *BE*, and equal to it

the pentagon in *G.*

nd *HK* parallel to *BE*

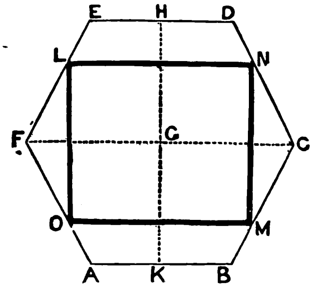
GL parallel to *HK*; then
uare, inscribed in the given

r **hexagon** *ABCDEF.*

FC by a perpendicular
ne of bisection cut the

as *FGH*, *CGH* (**Pr. 4**),

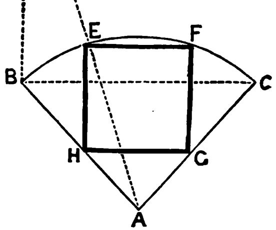
and produce the lines of bisection to meet the hexagon in *L*, *M*, *N*, *O*.



3. Join *LN*, *NM*, *MO*, and *OL* by straight lines, and *the figure LMNO is the required square, inscribed in the given hexagon ABCDEF.*

Problem 103.

To inscribe a **square** *within a given* **quadrant** *ABC, two of its corners being in the arc.*



1. Draw the chord *BC*, and at one of the extremities, say *B*, draw *BD* perpendicular and equal to it (**Pr. 2**).

2. Draw the line *DA*, cutting the arc *BC* in *E*.

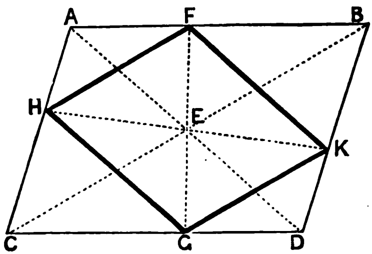
3. From *C*, cut off *CF*, equal to *BE*, and draw the chord *EF*. *EF is a side of the required square.* Complete the square (**Pr. 34**), and *EFGH will be the required square, inscribed within the given quadrant ABC.*

NOTE.—The same method is to be observed in inscribing a square in *any* sector of a circle (acute-angled or obtuse-angled).

Problem 104.

To inscribe a **four-sided equilateral figure** *in any given* **parallelogram** *ABCD.*

1. Draw the diagonals *AD*, *BC*, cutting each other in *E*.



2. Bisect any two of the adjacent angles at *E* (**Pr. 4**), by lines cutting the sides of the parallelogram in *F*, *G*, *H*, *K*.

3. Join *HF*, *FK*, &c., and *GHFK will be a four-sided equilateral figure, inscribed in a given parallelogram ABCD.*

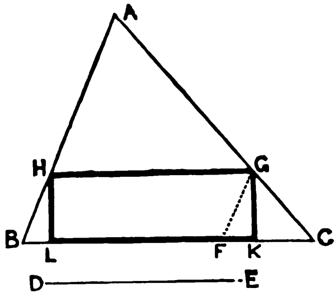
Problem 105.

To inscribe a **rectangle** *in a given* **triangle** *ABC, having a* **side** *equal to a given* **line** *DE.*

1. On *BC*, mark off *BF* equal to *DE*.

2. Through *F*, draw *FG* parallel to *AB*; and through *G*, draw *GH* parallel to *BC* (**Pr. 9**).

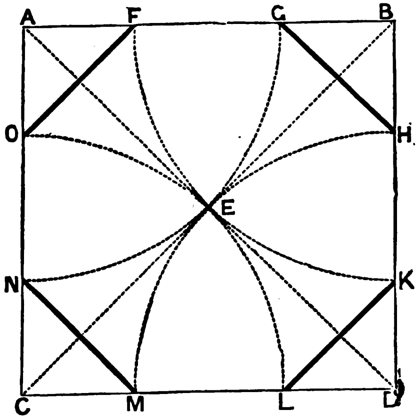
3. From G and H, draw GK and HL perpendicular to the base BC (**Pr. 3**); then *$HGKL$ is the required rectangle, and it is inscribed in the given triangle ABC.*



Problem 106.

To inscribe an **octagon** *in a given* **square** *$ABCD$.*

1. Draw the diagonals AD, BC, intersecting each other in E.

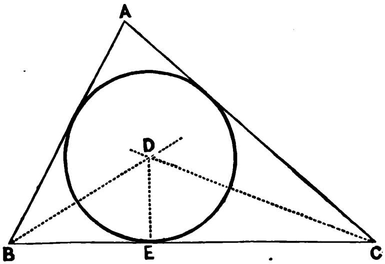


2. From A, B, C, and D, *with CE as radius,* describe quadrants cutting the sides of the square in F, G, H, K, L, M, N, O.
3. Join these points, and *the required octagon will be inscribed in the given square $ABCD$.*

Problem 107.

To inscribe a **circle** *within any given* **triangle** *ABC.*

1. Bisect *any* two of the angles, as *B* and *C* (**Pr. 4**), and let the bisecting lines be produced and meet at *D*, the centre of the triangle.

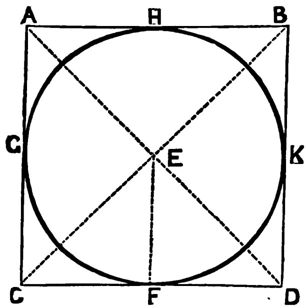


2. From *D* draw a perpendicular *DE*, to *any* side of the triangle (**Pr. 3**).
3. From centre *D*, *with radius DE, inscribe the required circle, which will be tangential to each side of the given triangle ABC.*

Problem 108.

To inscribe a **circle** *within a given* **square** *ABCD.*

1. Draw the diagonals *AD*, *BC*, cutting each other in *E*.

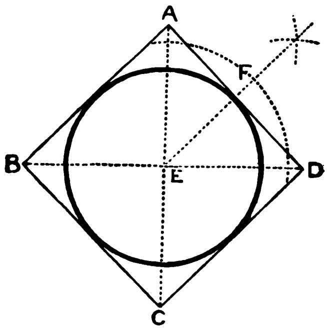


2. From *E*, draw *EF* perpendicular to *CD* (**Pr. 3**).
3. From *E* as centre, *with radius EF*, draw a circle *FGHK*, *which will be inscribed in the given square ABCD.*

Problem 109.

To inscribe a **circle** *in a given* **rhombus** *ABCD.*

1. Draw the diagonals *AC*, *BD*, cutting each other in *E*.

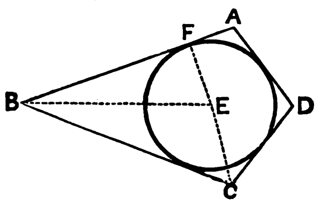


2. From *E*, draw a line perpendicular to any side *AD*, cutting it in *F* (**Pr.** 3).
3. From centre *E*, *with radius EF*, *inscribe the required circle in the given rhombus ABCD.*

Problem 110.

To inscribe a **circle** *in a given* **trapezium** *ABCD, which has its* **adjacent pairs of sides equal.**

1. Bisect any two of its adjacent angles *ABC*, *BCD* by lines meeting in *E* (**Pr.** 4).

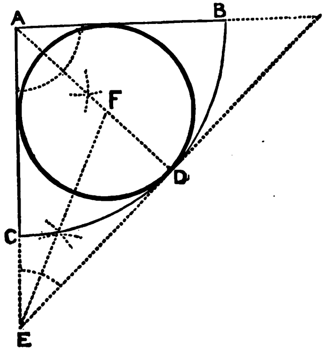


2. From point *E*, draw *EF* perpendicular to one of the sides *AB* (**Pr.** 3).
3. From centre *E*, *with radius EF, inscribe the required circle within the given trapezium ABCD.*

Problem 111.

To inscribe a **circle** *in a given* **quadrant** *ABC.*

1. Bisect the angle at *A*, the bisecting line cutting the arc in *D* (**Pr. 4**).



2. At point *D*, draw a tangent to the arc (**Pr. 54**), meeting one or both of the sides of the angle produced, as *AC* in *E*.

3. Bisect the angle at *E* (**Pr. 4**), the line of bisection cutting *AD* in *F*, the centre. *The circle described from F, with radius FA, will be the required circle, inscribed in the given quadrant ABC.*

NOTE.—The same method is to be observed in inscribing a circle in *any* sector of a circle (acute-angled or obtuse-angled).

Problem 112.—The Trefoil.

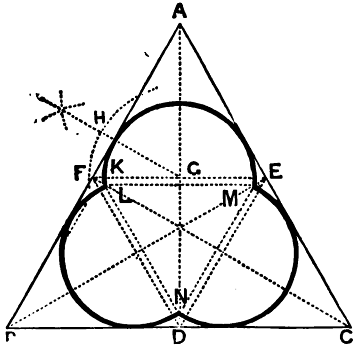
To inscribe **three equal semicircles,** *having their adjacent diameters equal, within a given* **equilateral triangle** *ABC.*

1. Bisect the angles at *A*, *B*, and *C* (**Pr. 4**), and draw the lines of bisection to meet the sides in *D*, *E*, *F*.

2. Join *DE*, *EF*, *FD;* and from *G*, in *EF*, draw *GH* perpendicular to *AB* (**Pr. 3**).

3. From *G*, *with GH as radius*, describe an arc, meeting *EF* in *K*.

4. Draw a line from K parallel to AB, cutting FC in L, and draw LM parallel to FE (**Pr. 8**).

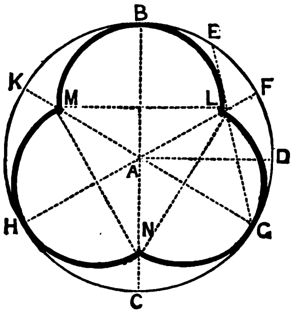


5. On LM describe an equilateral triangle LMN (**Pr. 18**), cutting AD in N.
6. On LM, MN, and LN as diameters, *describe the three required semicircles within the given equilateral triangle ABC.*

Problem 113.

To inscribe **three equal semicircles,** *having adjacent diameters, within a given* **circle** *A.*

1. Find the centre A of the given circle (**Pr. 45**).



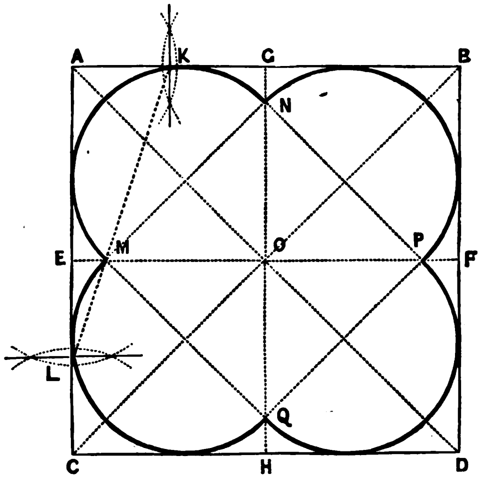
2. Draw *any* diameter BC, and the radius AD perpendicular to it.

3. Trisect the arc *BD* in *E* and *F* (**Pr. 5**).
4. On the other side of *D*, cut off *DG* equal to *DF*, and draw the diameters *FH*, *GK*.
5. Join *EG*, cutting the diameter *FH* in *L*.
6. From the centre *A*, *at the distance AL*, cut off *M* and *N* on the diameters *KG* and *BC*.
7. Join *LM*, *MN*, *NL*; then *LM*, *MN*, *and NL are the adjacent diameters of the three required semicircles to be inscribed within the given circle A.*

Problem 114—The Quatrefoil.

To inscribe **four equal semicircles**, *having their diameters adjacent, within a given* **square** *ABCD*, **each touching two sides of the square.**

1. Draw the diagonals *AD*, *BC*, also the diameters *EF*, *GH*.

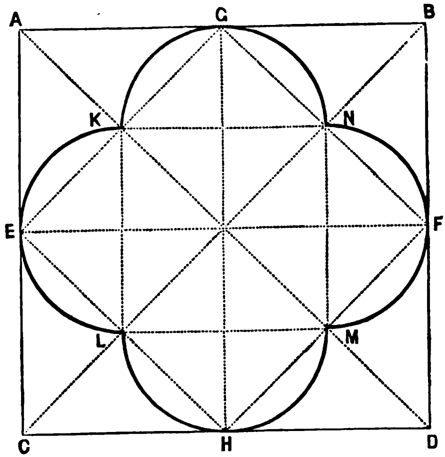


2. Bisect *AG* and *EC* in the points *K* and *L* (**Pr. 1**).
3. Join *KL*, cutting *EF* at *M*.
4. With centre *O*, *and radius OM*, mark off *N*, *P*, *Q*.
5. Join *MN*, *NP*, *PQ*, *QM*, *which are the diameters on which to describe the four required semicircles within the given square ABCD.*

Problem 115.

To inscribe **four equal semicircles,** *having their diameters adjacent, within a given* **square** *ABCD,* **each touching one side of the square.**

1. Draw the diagonals *AD* and *BC,* also the diameters *EF* and *GH.*



2. Draw the remaining diagonals of each of the smaller squares—viz., *GE, EH, HF,* and *FG,* intercepting the former diagonals in *K, L, M,* and *N.*

3. Join these points by lines *KL, LM, MN, NK, which are the diameters on which to describe the four required semicircles within the given square ABCD.*

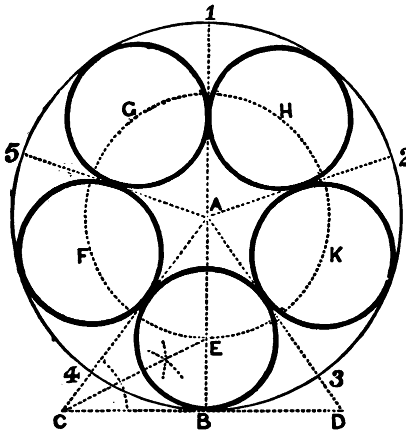
Problem 116.

To inscribe **any number of equal circles** (in this case five) *in a given* **circle** *A.*

General Method.

1. Divide the circumference into the *same* number of equal parts as there are required circles to be inscribed (in

this case *five*), and draw radii to each point of division, as *A* 1, *A* 2, &c. *In each of these five sectors a circle is to be inscribed.*



2. Bisect one of them, as 4 *A* 3, by radius *AB* (**Pr. 4**), and draw a tangent at B (**Pr. 54**), cutting *A* 4 and *A* 3 produced in *C* and *D*.

3. Find *E*, the centre of the triangle *ACD* (**Pr. 22**), and inscribe a circle in it.

4. From *A* as centre, *with AE as radius*, describe the inner circle.

5. On this circle, mark off from *E*, the centres for the four other circles, as *F*, *G*, *H*, *K*; and inscribe them. *There will then be five equal circles, inscribed in the given circle A.*

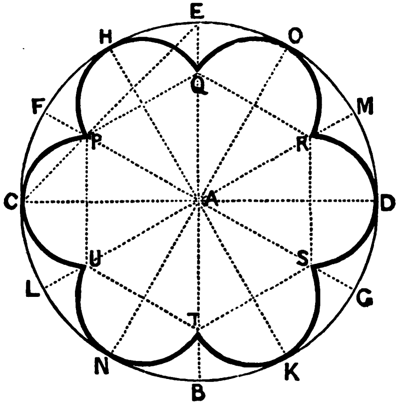
Problem 117.

To inscribe **any number of equal semicircles** (in this case six) *in a given* **circle** *A*.

1. Draw the diameters *EB* and *CD* at right angles to each other.

2. Divide the circumference into *twice* as many equal parts as there are to be semicircles (in this case twelve equal parts) by trisecting each right angle (**Pr. 5**).

3. Draw the diameters *FG, HK, LM, NO;* and *no matter how many semicircles are required,* join *EC,* and where it cuts the *next* diameter, as *FG,* we obtain a point *P,* which is the extremity of the diameter of one of the semicircles.



4. From *A,* mark off on each *alternate* diameter, the distances *AQ, AR, AS, AT, AU,* equal to *AP.*

5. Join *PQ, QR, RS, ST, TU,* and *UP. These are the diameters on which to inscribe the required semicircles in the given circle A.*

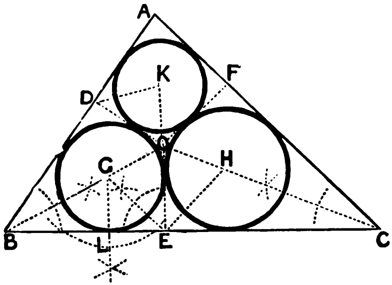
Problem 118.

To inscribe **three circles** *in any given triangle ABC, each touching two others, and* **two sides of the triangle.**

1. Find the centre *O* of the triangle *ABC* by bisecting two angles *B* and *C* (**Pr. 22**), and from *O,* draw a perpendicular to each side of the triangle (**Pr. 3**) meeting *AB* in *D, BC* in *E,* and *AC* in *F.*

2. Bisect two adjacent angles of each of three quadrilaterals thus formed, the bisecting lines meeting in *G, H, K,* the centres of the three required circles, *the radius of each circle being found by drawing a perpendicular from*

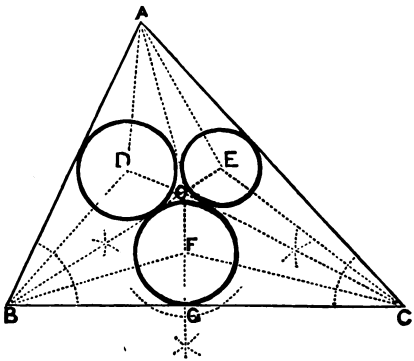
each centre to one side of each quadrilateral, as GL to BE (**Pr.** 3).



3. *Inscribe the required circles G, H, and K in the given triangle ABC.*

Problem 119.

To inscribe **three circles** *in any given* **triangle** *ABC, each touching the other two, and* **one side of the triangle.**



1. Find the centre *O* of the triangle *ABC* by bisecting the angles at *B* and *C* (**Pr.** 22), and draw *AO*.

2. Inscribe a circle in each of the triangles thus formed, by bisecting two angles of each, the bisecting lines meeting

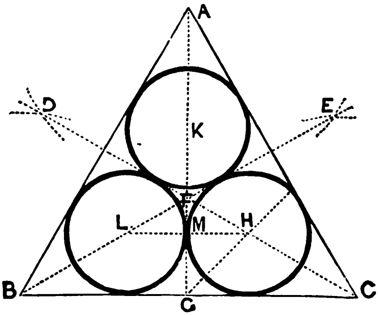
in D, E, F, the radius of each circle being found by drawing a perpendicular from the centre to one of the sides of each triangle, as FG to BC (**Pr. 22**).

3. *Inscribe the required circles D, E, and F in the given triangle ABC.*

Problem 120.

To inscribe **three equal circles** *within a given* **equilateral triangle** *ABC, touching each other, and* **two sides of the triangle.**

1. From *A* and *B*, and *A* and *C*, describe arcs intersecting in *D* and *E*.



2. Draw the lines *BE* and *CD*, cutting the centre of the triangle in *F*.
3. Draw the line *AG*, and bisect the angle *FGC* (**Pr. 4**), the line of bisection cutting *CD* in *H*.
4. From *F*, set off the distance *FH* on the lines *AG* and *BE*, in the points *K* and *L*; then *H, K, L will be the three centres of the required circles.*
5. Draw the line *LH*, and *HM will be the radius of the required circles, to be inscribed in the given equilateral triangle ABC.*

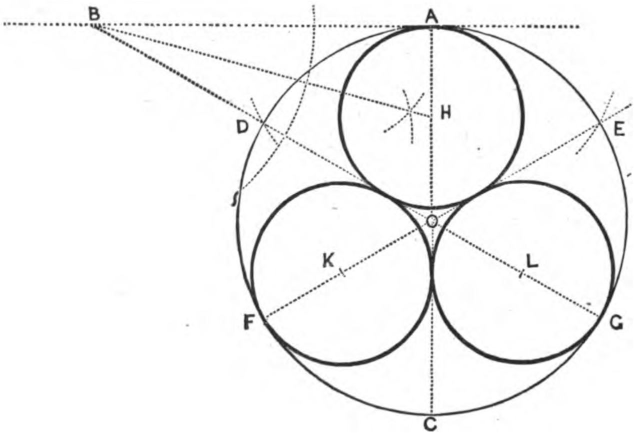
Problem 121.

To inscribe **three equal circles** *in a given* **circle** *O.*

1. At any point *A*, draw a tangent *AB* (**Pr. 54**), and *AC* at right angles to it (**Pr. 2**).

2. From A, *with radius* AO, cut the circumference in D and E.

3. From D and E, draw lines through O, cutting the circumference in G and F, and the tangent in the point B.



4. Bisect the angle ABD (**Pr. 4**), and produce the line of bisection until it meets AC in H.

5. From O, *with radius* OH, cut the lines EF and DG in the points K and L.

6. From H, K, and L, *with radius* HA, *describe the three required circles, each of which will touch the other two, and the given outer circle* O.

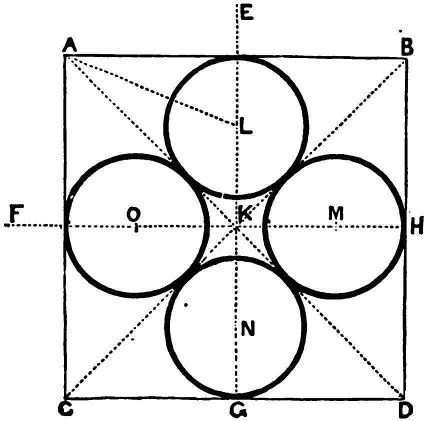
Problem 122.

To inscribe **four equal circles** *in a given* **square** *ABCD, touching each other, and* **one side only** *of the square.*

1. Draw the diagonals AD, BC. With centres A, B, and C, *and any radius*, describe arcs at E and F.

2. From E and F, draw the diameters EG, FH.

3. The diagonals divide the square into four equal triangles, viz., *AKB*, *BKD*, *CKD*, and *AKC*. We have therefore only to describe a circle in each (**Pr. 107**).



4. In the triangle *AKB* the angle *AKB* is already bisected by *KE*; by bisecting one of the other angles, say *KAB*, by the line *AL* (**Pr. 4**), we obtain point *L*, the centre of one of the circles.

5. With centre *K*, *and radius KL*, mark off the points *M*, *N*, *O*. Then with centres *L*, *M*, *N*, *O*, *and radius LE*, *describe the four required circles in the given square ABCD.*

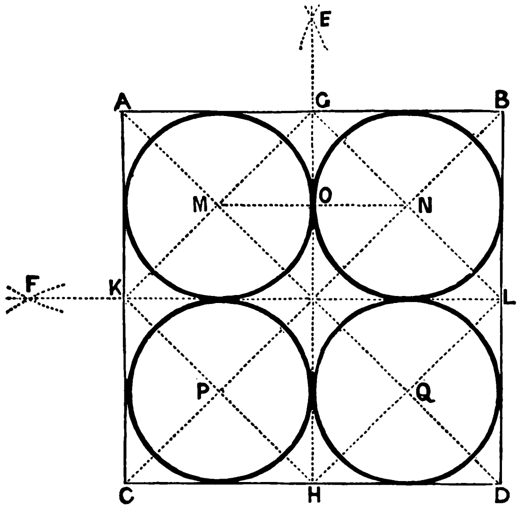
Problem 123.

To inscribe **four equal circles** *within a given* **square** *ABCD, touching each other, and each circle also* **to touch two sides** *of the given square.*

1. Draw the diagonals *AD*, *BC*. With centres *A*, *B*, and *C*, *and any radius*, describe arcs at *E* and *F*.

2. From *E* and *F*, draw the diameters *GH* and *KL*.

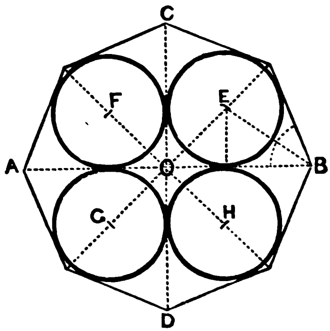
3. Join *KG*, *GL*, *LH*, and *HK*. Also join *MN*.



4. With centres *M*, *N*, *P*, and *Q*, *and radius MO, describe the four required circles within the given square ABCD.*

Problem 124.

To inscribe **four equal circles** *in a given* **octagon.**



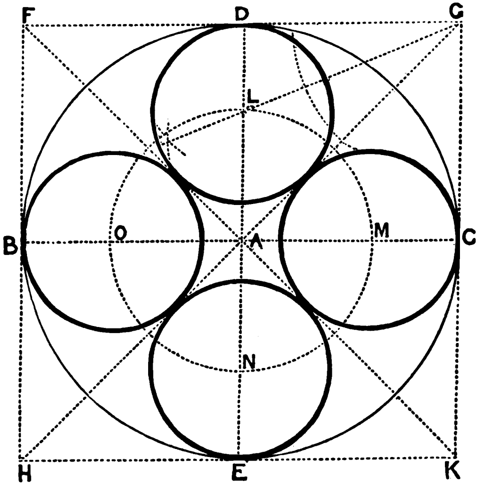
1. Draw any two diagonals at right angles to each other, as

AB, *CD*, intersecting each other in the centre *O*. These divide the given octagon into four equal trapezia.

2. Find the centre of each, as *E*, *F*, *G*, *H*, and inscribe a circle in each trapezium (**Pr. 110**). *The required four equal circles will then be inscribed in the given octagon.*

Problem 125.

To inscribe **four equal circles** *in a given* **circle** *A*.

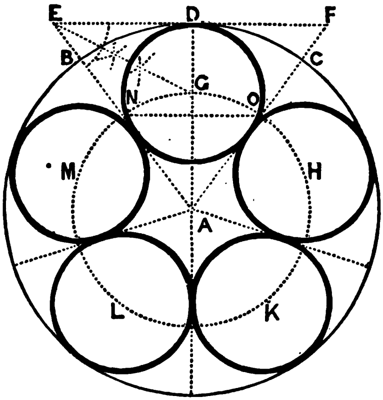


1. Draw the diameters *BC* and *DE* at right angles to each other.
2. From *B*, *C*, *D*, and *E*, describe arcs cutting each other in *F*, *G*, *H*, *K*.
3. Join these points, and a square will be described about the circle *A*.
4. Draw the diagonals *FK* and *HG*.
5. Bisect the angle *DGA* (**Pr. 4**), and produce the line of bisection until it cuts *DE* in *L*.
6. From *A*, *with radius AL*, describe a circle cutting the lines *BC* and *DE* in *M*, *N*, *O*.
7. From centres *L*, *M*, *N*, *O*, *with radius LD*, *describe the four required circles within the given circle A*.

Problem 126.

To inscribe **five equal circles** *in a given* **circle** *A.*

1. Divide the circumference into five equal parts, as in the case of inscribing a pentagon (**Pr. 63**).



2. From the centre *A*, draw lines through two divisions, as *B* and *C*, and produce them.

3. Bisect the angle *BAC* (**Pr. 4**), and draw *AD*, touching the circumference of the given circle in *D*.

4. At *D*, draw a tangent to the circle (**Pr. 54**), cutting *AB* and *AC* produced, and completing the triangle *EAF*.

5. Inscribe a circle in this triangle (**Pr. 107**), having its centre at *G*.

6. From *A*, ***with AG as radius***, inscribe the circle *GHKLM*, cutting *AE* and *AF* in *N* and *O*.

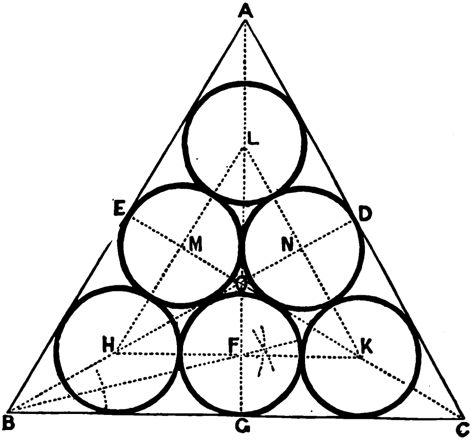
7. From *A*, ***with the line NO as radius***, cut the circumference of the inner circle in *H*, *K*, *L*, *M*.

8. From those points, *with radius DG, describe the remaining four circles within the given circle A, to complete the figure.*

Problem 127.

To inscribe **six equal circles** *within a given* **equilateral triangle.** *ABC.*

1. Draw the lines *BD*, *CE*, and *AG*, bisecting the angles and sides of the given triangle, and cutting each other in *O*.



2. Bisect the angle *OBG* (**Pr. 4**), and the point *F*, where the line of bisection cuts *AG*, will be the centre of one of the isosceles triangles into which the equilateral triangle has been divided.
3. Through *F*, draw *HK* parallel to *BC* (**Pr. 9**); and from *H* and *K*, draw *HL* and *KL* parallel to *AB* and *AC*, and cutting *CE* and *BD* in *M* and *N*.
4. From points *F*, *H*, *K*, *L*, *M*, *N*, *with radius FG, describe the six required circles within the given equilateral triangle ABC.*

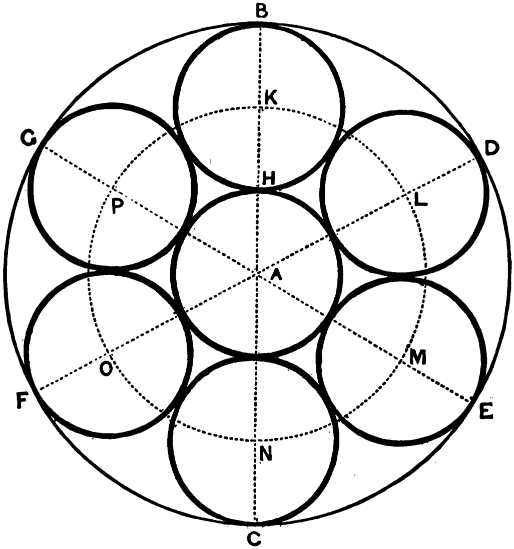
Problem 128.

To inscribe **seven equal circles** *in a given* **circle** *A.*

1. Draw a diameter *BC*, and from the point *B*, *with the radius of the circle*, divide the circumference into six equal parts, in *D*, *E*, *C*, &c., and draw the radii.

2. Divide one of the radii, as *AB*, into three equal parts, in the points *H*, *K*.

3. From *A*, *with radius AH*, describe the central circle.



4. From *A*, *with radius AK*, describe a circle which, cutting the radii, will give the points *L*, *M*, *N*, *O*, *P*.

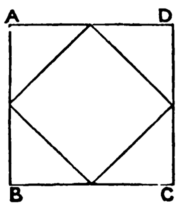
5. From points *K*, *L*, *M*, *N*, *O*, *P*, *with radius AH*, describe the six circles, which, with the central circle, *constitute the seven required circles within the given circle A*.

Section VIII.—DESCRIBED FIGURES.

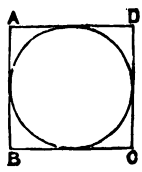
DEFINITIONS.

1. **Described figures.** Described figures are either *rectilineal* or *circular.*

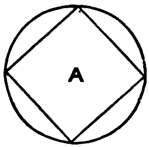
(*a.*) A rectilineal figure is said to be *described about* another rectilineal figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described. *Ex. ABCD*—



(*b.*) A rectilineal figure is said to be *described about* a circle, when each side of the circumscribed figure touches the circumference of the circle. *Ex. ABCD*—



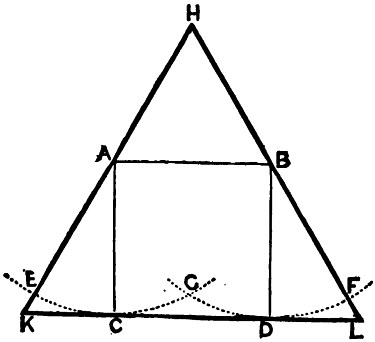
NOTE.—A *circle* is said to be *described about* a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described. *Ex. A*—



Problem 129.

To describe an **equilateral triangle** *about a given* **square** *ABCD.*

1. From points *A* and *B*, *with AC as radius*, describe arcs cutting each other in *G*.



2. From *G* as centre, *with the same radius*, cut these arcs in *E* and *F*.

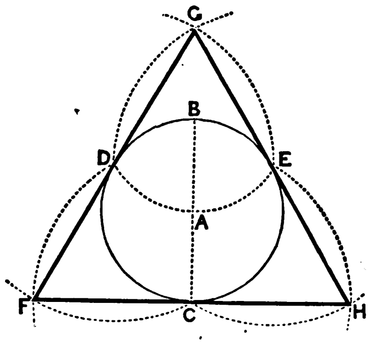
3. Join *EA* and *FB*, and produce them to meet in *H*.

4. Produce *CD* until it cuts the lines *HE* and *HF* produced in *K* and *L*; *then HKL is the required equilateral triangle described about the given square ABCD.*

Problem 130.

To describe an **equilateral triangle** *about a given* **circle** *A.*

1. Draw a diameter *BC.*

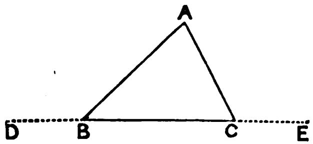


2. From *B, with radius BA,* cut the circumference in *D* and *E.*
3. From *D, E,* and *C* as centres, *with DE as radius,* describe arcs intercepting in *G, F,* and *H.*
4. Join *GF, FH,* and *HG;* then *FGH is the required equilateral triangle described about the given circle A.*

Problem 131.

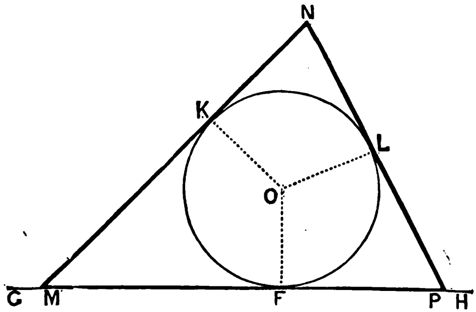
To describe a **triangle** *about a given* **circle** *O, having* **angles equal** *to those of a* **given triangle** *ABC.*

1. Produce *any* side of the triangle, as *BC,* both ways to *D* and *E.*



2. Draw *any* radius *OF,* and draw *GH* as a tangent through *F* (**Pr. 54**).

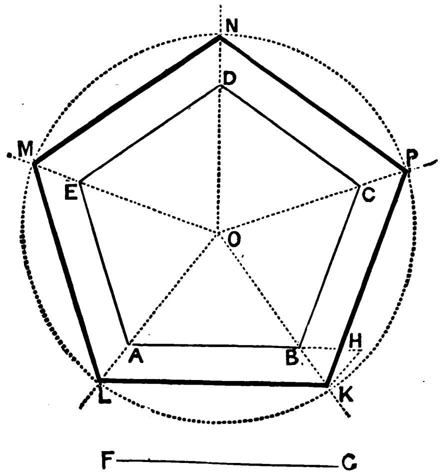
3. Make angle *FOK* equal to angle *ABD*, and angle *FOL* equal to angle *ACE* (**Pr. 10**).



4. Through *K*, draw *MN* at right angles to *KO*, and through *L*, draw *NP* at right angles to *LO* (**Pr. 2**) ; then *the triangle MNH is described about the given circle O, and has its angles equal to those of the given triangle ABC.*

Problem 132.

To describe a **pentagon** *about a given* **pentagon** *ABCDE, having its* **sides parallel** *to it, and equal to a given* **straight line** *FG.*



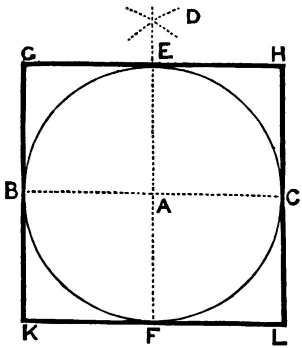
1. **Find** the centre of the pentagon by bisecting two adjacent angles (**Pr. 4**).

2. From the centre O, draw the five radii, and produce them *indefinitely.*
3. Produce *one* of the sides, as AB, *until it is equal to* FG; viz. AH.
4. From H, draw a line parallel to OA, cutting the radius OB in K.
5. From O, *with radius* OK, describe a circle cutting the produced radii in L, M, N, P.
6. Join KL, LM, &c., by straight lines, and *a pentagon KLMNP will be described about a given pentagon ABCDE, and having its sides equal to the given line FG.*

Problem 133.

To describe a **square** *about a given* **circle** *A.*

1. Draw the diameter BC. With centres B and C, *and any radius*, describe arcs at D. From D, draw the diameter EF.



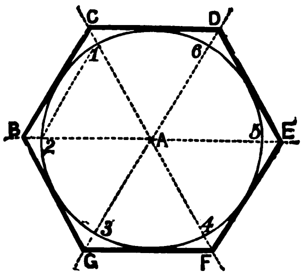
2. With centres B, E, C, and F, *and radius* BA, describe arcs cutting at G, H, K, L.
3. Join GH, HL, LK, and KG, and $GHLK$ *is the required square described about the given circle A.*

Problem 134.

To construct **any regular polygon** (*say a* **hexagon**) *about a given* **circle** *A.*

1. Divide the circumference into *as many equal parts* as the polygon is to have sides—six (**Pr. 64**).

2. Draw radii to these points of division, and produce them beyond the circumference.



3. Join 1 2, and draw *BC* parallel to 1 2, and tangential to the circle (**Pr. 54**).

4. Take the distance from the centre of the circle to *C*, and mark off from the centre points *D*, *E*, &c.

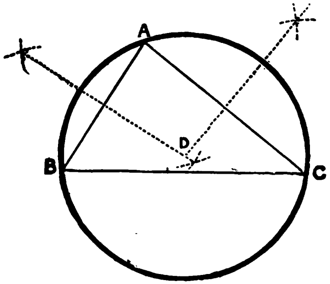
5. Join *CD*, *DE*, &c., by straight lines, and *a regular hexagon BCDEFG will be constructed about a given circle A.*

Problem 135.

To describe a **circle** *about a given* **triangle** *ABC.*

1. Bisect *any* two of its sides *AB*, *AC* by lines cutting in *D* (**Pr. 1**).

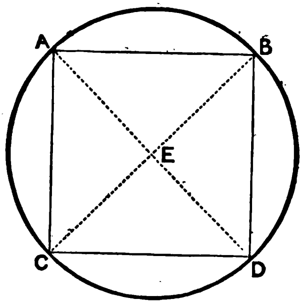
2. From *D* as centre, *with DA, DB, or DC as radius,* describe a circle; then *ABC is the required circle, described about the given triangle ABC.*



Problem 136.

To describe a **circle** *about a given* **square** *ABCD.*

1. Draw the diagonals *AD, BC,* intersecting each other in *E.*



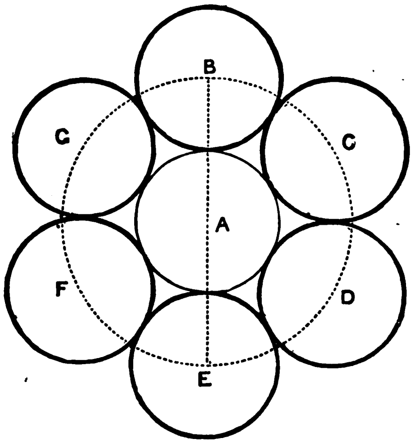
2. From *E* as centre, *with radius EA, draw a circle ABDC, which will be described about the given square ABCD.*

Problem 137.

To describe **six equal circles** *about, and equal to, a given* **circle** *A, touching each other and the given circle.*

1. From the centre *A* of the given circle, *and with its diameter as radius,* describe the circle *BCDEFG.*

2. Draw the diameter BAE, and from B, *with the radius of the given circle*, describe a circle touching it.



3. From B, mark off the other centres C, D, E, &c.

4. From these points C, D, E, &c., *with the radius of the given circle*, describe the remaining five circles, *which will touch each other, and the given circle* A.

Problem 138.

To describe **any number of equal circles** *about a given* **circle** A, *each touching two others, and the given circle.* (Say eight in this case).

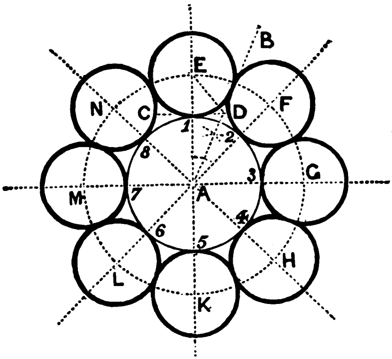
1. Divide the circumference into *eight* equal parts, and draw produced radii through the points of division.

2. Bisect *one* of the angles at the centre, as $1A2$ (**Pr. 4**), by a line AB, and draw a tangent CD at point 1 (**Pr. 54**), cutting the bisecting line in D.

3. Bisect the *obtuse* angle CDB (**Pr. 4**) by a line, which, produced, cuts the radius $A1$ produced in E.

4. From A as centre, *with radius* AE, describe the outer circle, cutting the produced radii in F, G, H, &c.

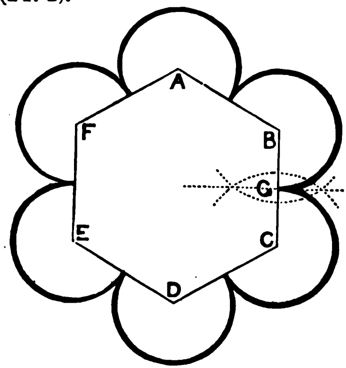
5. From these points, *E, F, G,* &c, *with E* 1 *as radius*, describe the eight circles, *each of which will touch two circles, and the given circle A.*



Problem 139.

To construct a **foiled figure** *about any given regular* **polygon** (say a **hexagon.**)

1. *ABCDEF* is the given hexagon. Bisect *one* side *BC* in *G* (**Pr. 1**).



2. With the several angles of the polygon as centres, *and radius GB*, describe the six arcs, *which form the required hexafoil about the given hexagon ABCDEF.*

Section IX.—PROPORTIONALS.

DEFINITIONS.

1. **Ratio** is the relation that one quantity bears to another *of the same kind*, in respect of magnitude, *i.e.*, by considering what multiple, part, or parts one is of the other. Thus, in comparing 6 with 3, we find that it has a certain magnitude with respect to 3, that is, it contains it *twice;* but in comparing it with 2, we find that it has a different relative magnitude, for it contains it *three times.*

Note 1.—The ratio of any two quantities (of the same kind) depends therefore on their *relative,* and not their *absolute,* magnitudes.

Note 2.—A "part" must be understood to mean any *aliquot* part (not *any* part).

Note 3.—The ratio of a to b is usually represented thus— $a : b$, or sometimes $\frac{a}{b}$.

2. "**Proportion** is the similitude of ratios" (**Euc. V., Def. 8**). Thus, four quantities are said to be *proportionals* when the first is the same multiple, part, or parts of the second that the third is of the fourth. This is usually expressed by saying a is to b as c is to d, and is thus represented, $a : b :: c : d$, or sometimes $a : b = c : d$.

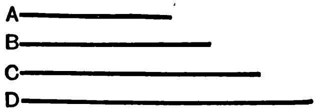
Note 1.—The last term (d) is called the *fourth* proportional.

Note 2.—The quantities a and d are called the *extremes,* and b and c the *means.*

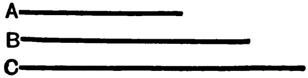
Note 3.—When four quantities are proportionals, the product of the extremes is equal to the product of the means, *i.e.,* $ad = bc$.

Note 4.—When the two means are the *same* quantity—as $a : b :: b : c$—the last term (c) is called the *third* proportional, and the middle term (b) is called a *mean* proportional.

3. A **proportional** in *Practical Geometry* is a line which bears some fixed ratio to one or more given lines. Thus, the four straight lines *A, B, C, D* are proportionals, *D* being the fourth proportional *greater*, and *A* the fourth proportional *less*, to the lines *A, B, C;* and *B, C, D,* respectively—



Again, the three straight lines *A, B, C* are proportionals, *C* being the third proportional *greater*, and *A* the third proportional *less* to the lines *A, B;* and *B, C,* respectively—

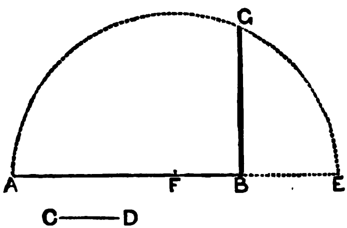


NOTE.—*B* is a *mean proportional* between the two lines *A* and *C.*

Problem 140.

To find a **mean proportional** *between* **two** *given lines AB and CD.*

1. Produce *AB* to *E,* and make *BE* equal to *CD.*



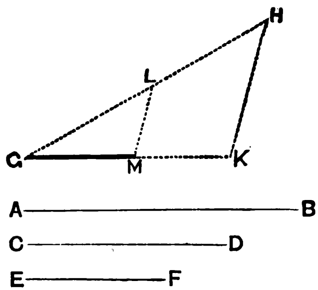
2. Bisect *AE* in *F* (**Pr. 1**). With centre *F, and radius FA,* describe the semicircle.

3. From *B,* raise a perpendicular *BG* (**Pr. 2**) to meet the semicircle. *Then BG is the required mean proportional between the two given lines AB and CD.*

Problem 141.

To find a **fourth proportional** *to* **three** *given straight* **lines,** *AB, CD, and EF, when the required line is* **less** *than any of the given lines.*

1. Make *GH* equal to *AB*, and draw *GK* equal to *CD*, making *any* angle with *GH*.



2. Join *HK*, and from *G*, on the line *GH*, cut off *GL* equal to *EF*.

3. Through point *L*, draw *LM* parallel to *HK* (**Pr. 9**). *Then GM is the required fourth proportional to the three given straight lines AB, CD, EF, and less than any of them.*

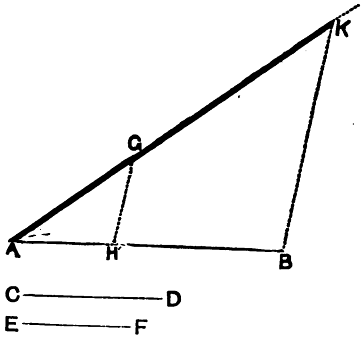
Problem 142.

To find a **fourth proportional** *to* **three** *given straight* **lines** *AB, CD, and EF, when the required line is* **greater** *than any of the given lines.*

1. At *A* in *AB*, draw a line *AG* equal to *CD*, and at *any* angle with *AB*.

2. From *A*, set off *AH* equal to *EF*, and join *GH*.

3. Through *B*, draw *BK* parallel to *GH* (**Pr.** 9), cutting *AG* produced in *K*. *Then AK is the required fourth pro-*

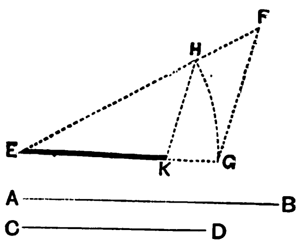


portional to the given lines AB, CD, and EF, and greater than any of them.

Problem 143.

To find a **third proportional** *to* **two** *given straight* **lines** *AB and CD, when the required line is* **less** *than either of the given lines.*

1. Make *EF* equal to *AB*, and draw *EG* equal to *CD*, making *any* angle with *EF*.

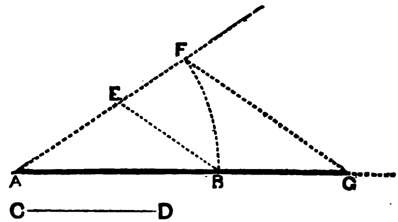


2. Join *FG*, and from *E*, *with radius EG*, cut *EF* in *H*.
3. Draw *HK* parallel to *FG* (**Pr.** 9). *Then EK is the required third proportional to the given straight lines AB and CD, and less than either of them.*

Problem 144.

To find a **third proportional** *to* **two** *given straight* **lines** *AB and CD, when the required line is* **greater** *than either of the given lines.*

1. At A in AB, draw a line AE equal to CD, and at *any* angle with AB.

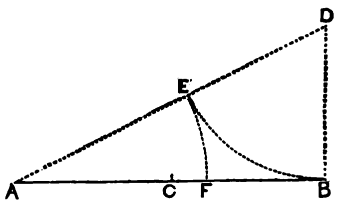


2. Join BE, and produce AE and AB *indefinitely.*
3. From A as centre, *with AB as radius,* describe an arc cutting AE produced in F.
4. Through F, draw FG parallel to EB (**Pr. 9**), cutting AB produced in G. *Then AG is the required third proportional to the given lines AB and CD, and greater than either of them.*

Problem 145.

To **divide** *a given straight* **line** *AB into* **extreme** *and* **mean proportion.**

1. Bisect AB in C (**Pr. 1**), and from one extremity, say B, erect a perpendicular BD equal to BC.



2. Join AD, and from the centre D, *with radius DB,* describe the arc BE.

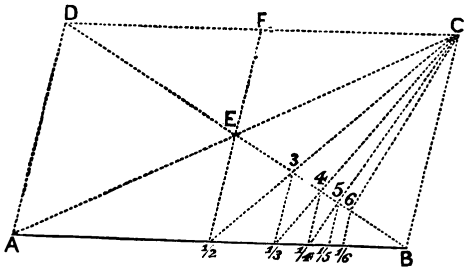
3. From centre *A, and radius AE,* describe the arc *EF. Then the given line AB is divided in extreme and mean proportion in the point F.*

NOTE.—" A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment as the greater segment is to the less " (**Euc. VI., Def. 3**).

Problem 146.

To **divide** *a given straight* **line** *AB* **successively** *into its half, third, fourth, fifth, &c.*

1. On *AB,* construct *any* parallelogram *ABCD,* and draw the diagonals *AC, BD,* intercepting each other in *E.*



2. Draw line $F\frac{1}{2}$ parallel to *AD* (**Pr. 9**) cutting *AB* in $\frac{1}{2}$.

3. Draw line $C\frac{1}{2}$, cutting *BD* in 3, and draw line $3\frac{1}{3}$ parallel to *AD,* cutting *AB* in $\frac{1}{3}$.

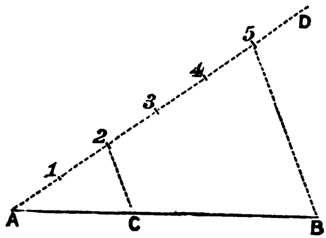
4. Draw line $C\frac{1}{3}$, cutting *BD* in 4, and through 4 draw line $4\frac{1}{4}$ parallel to *AD,* cutting *AB* in $\frac{1}{4}$, &c., &c. *The divisions thus obtained are the required half, third, fourth, &c., of the given straight line AB.*

Problem 147.

To **divide** *any given straight* **line** *AB in the point C, so that AC : CB* : : **2 : 3.**

1. From point *A,* draw a straight line *AD* of *indefinite* length, and at *any* angle to *AB.*

2. On AD, mark off *any* two equal distances A 1, 1 2, and from 2, mark off three *similar* distances to 5; and join 5 B.

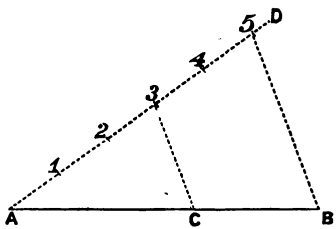


3. Through 2, draw 2 C parallel to 5 B (**Pr. 9**). *Then the given straight line* AB *is divided in the point* C, *so that* $AC : CB :: 2 : 3$.

Problem 148.

To **divide** *any given straight* **line** AB *in the point* C, *so that the* **whole** AB *is to one* **part** AC *as* 5 : 3.

1. From point A, draw a straight line AD of *indefinite* length, and at *any* angle to AB.



2. Take *any* five equal distances on AD, and join 5 B.

3. At point 3, draw a line 3 C parallel to 5 B (**Pr. 9**). *Then the given straight line* AB *is divided in the point* C, *so that* $AB : AC :: 5 : 3$.

Section X.—SIMILAR FIGURES.

DEFINITIONS.

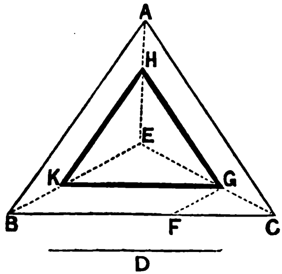
1. "**Similar rectilineal figures** are those which have their several angles equal, each to each, and the sides about the equal angles proportionals" **(Euc. VI., Def. 1).**

NOTE 1.—The following rectilineal figures are similar, viz. :— *All equilateral triangles, squares,* and *regular polygons of the same name.*

NOTE 2.—*Other* rectilineal figures, *e. g., triangles which are not equilateral, trapeziums,* and *irregular polygons,* can be made similar to given ones, as shown in the problems following.

Problem 149.

To **inscribe within** *and* **equidistant** *from the sides of a given* **triangle** *ABC, a* **similar triangle,** *one of whose sides is equal to a given* **line** *D.*



1. Bisect the angles of the given triangle **(Pr. 4)** by lines meeting in *E.*

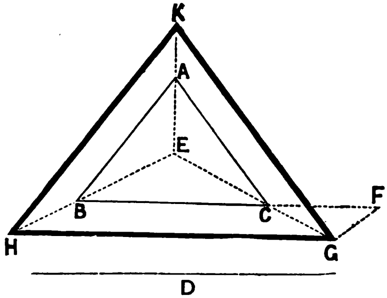
2. Make *BF* equal to *D*, and through *F* draw *FG* parallel to *BE* (**Pr. 9**).

3. Through *G*, draw lines parallel to the sides of the given triangle, cutting the bisecting lines in *H* and *K*.

4. Join *HK*; *then GHK is a similar triangle, inscribed within the given triangle ABC, and having its side KG equal to the given line D.*

Problem 150.

To **describe about** *and* **equidistant** *from the sides of a given* **triangle** *ABC, a* **similar triangle,** *one of whose sides is equal to a given* **line** *D.*



1. Bisect the angles of the given triangle (**Pr. 4**) by lines meeting in *E*.

2. Produce the side *BC* to *F*, making *BF* equal to the given line *D*.

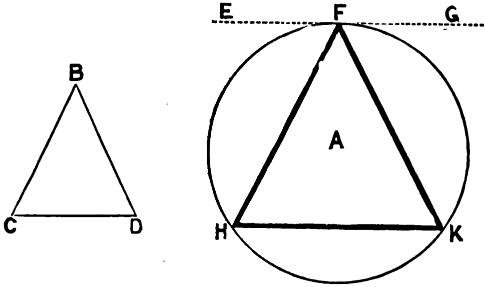
3. Through *F*, draw *FG* parallel to the bisecting line *BE*, cutting *EC* produced in *G*.

4. Through *G*, draw the line *GH* parallel to *BC* (**Pr. 9**), cutting *EB* produced in *H*.

5. Through *G* and *H*, draw the lines *GK* and *HK*, parallel to the sides of the given triangle *ABC*, meeting *EA* produced in *K*. *Then GHK is a similar triangle described about the given triangle ABC, and having its side HG equal to the given line D.*

Problem 151.

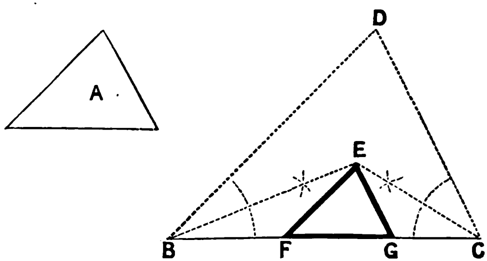
To inscribe a **triangle** *in a given* **circle** *A*, **similar** *to a given* **triangle** *BCD.*



1. Draw a tangent *EFG* at *any* point *F* in the circumference (**Pr. 54**).
2. From *F*, draw *FH*, making with *EF* an angle equal to *BCD* (**Pr. 10**), and meeting the circumference in *H*.
3. From *F*, draw *FK*, making with *FG* an angle equal to *BDC*, and meeting the circumference in *K*.
4. Join *HK*. *Then FHK is a triangle similar to the given triangle BCD, inscribed within the given circle A.*

Problem 152.

To construct a **triangle similar** *to a given* **triangle** *A, and having its* **perimeter** *equal to a given* **straight line** *BC.*



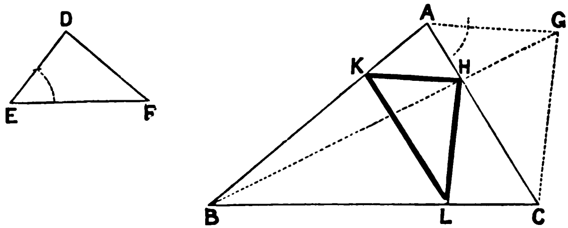
1. On *BC*, construct a triangle *BDC*, having its angles equal to those of the given triangle *A* (**Pr. 23**).

2. Bisect the angles at *B* and *C* (**Pr. 4**) by lines meeting in *E*.

3. Through *E*, draw *EF* and *EG* parallel to *DB* and *DC* (**Pr. 9**), meeting *BC* in *F* and *G*. *Then EFG is the required similar triangle, having its perimeter EF, FG, GE equal to the given straight line BC.*

Problem 153.

To inscribe a **triangle** *within a given* **triangle** *ABC, and* **similar** *to another given* **triangle** *DEF.*



1. On the side *AC* of the triangle *ABC*, construct a triangle *ACG* similar to the given triangle *DEF* by measuring the angles at *E* and *F*.

2. Join *BG*, cutting *AC* in *H*.

3. Through *H*, draw *HK* parallel to *AG* (**Pr. 9**), and *HL* parallel to *GC*, meeting *BC* in *L*.

4. Join *KL*. *Then HKL will be the required triangle inscribed within the given triangle ABC, and similar to the given triangle DEF.*

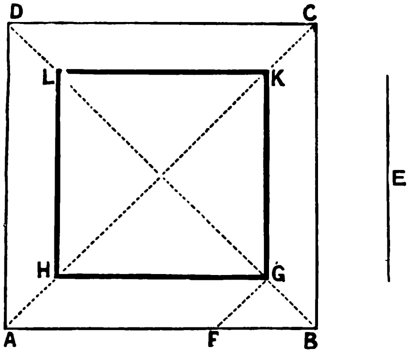
Problem 154.

To construct within a given **square** *ABCD, another* **square concentric** *with it, and having its* **side** *equal to a given* **line** *E.*

1. Draw the diagonals *AC, BD;* and on *AB* cut off *AF* equal to *E*.

2. Through *F*, draw *FG* parallel to *AC* (**Pr. 9**), and through *G*, draw *GH* parallel to *AB*.

3. Through *G* and *H*, draw *GK* and *HL* parallel to *BC* or *AD*.



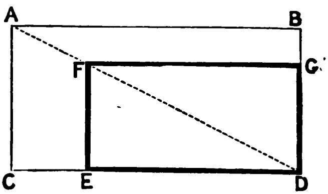
4. Join *LK*. *Then GHLK is the required concentric square, and having its side equal to the given line E.*

NOTE.—If it is required to describe a square *about* a given square, make *AB* produced equal to the required side, and proceed as in Problem 150.

Problem 155.

To construct a **rectangle, similar** *to a given* **rectangle** *ABCD, on a* **part** *ED of the* **side** *CD of the given rectangle.*

1. Draw the diagonal *AD*, and from *E*, draw a line *EF* parallel to *AC* or *BD* (**Pr. 9**), meeting *AD* in the point *F*.

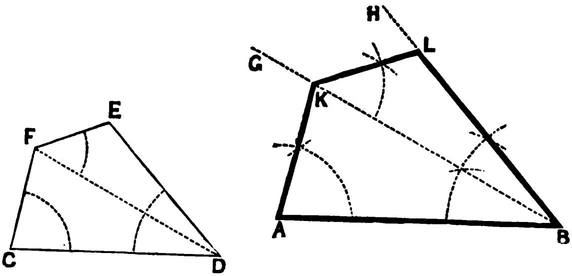


2. From *F*, draw a line *FG* parallel to *CD* (**Pr. 9**), meeting *BD* in the point *G*. *Then DEFG is the required rectangle, and is similar to the given rectangle ABCD.*

Problem 156.

To construct a **trapezium** *on a given* **line** *AB, which shall be* **similar** *to a given* **trapezium** *CDEF.*

1. At *B*, in the given line *AB*, make angle *ABG* equal to angle *CDF* (**Pr. 10**), and angle *ABH* equal to angle *CDE.*



2. Make angle *BAK* equal to angle *DCF.*

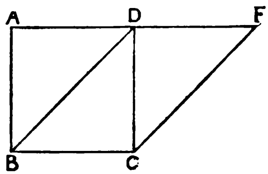
3. At *K*, make angle *BKL* equal to angle *DFE. Then AKLB will be the required trapezium, constructed on the given line AB, and similar to the given trapezium CDEF.*

NOTE.—By means of this Problem, *any* rectilineal figure may be constructed *similar* to another given rectilineal figure, either *greater* or *less*.

Section XI.—EQUIVALENT AREAS.

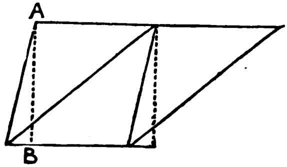
(*Before the student enters on the following problems he should thoroughly master the subjoined theorems.*)

(**A.**) "Parallelograms upon the *same* base, and between the same parallels, are equal to each other" (in area).—**Euc. I., 35.** *Ex. ABCD, DBCF*—

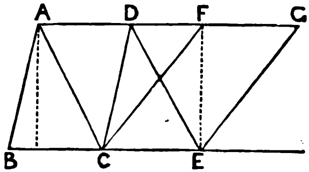


Note 1.—"Between the same parallels" means *having the same altitude.*

Note 2.—The altitude must always be *perpendicular* to the *base. Ex. AB*—

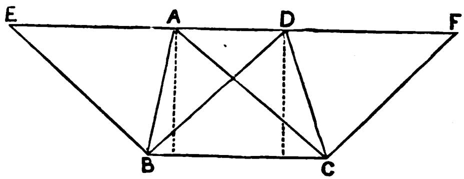


(**B.**) "Parallelograms upon *equal* bases and between the same parallels are equal to one another" (in area).—**Euc. I., 36.** *Ex. ABCD and ACED*, or *ABCD and FCEG*—



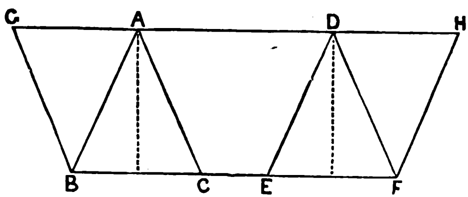
NOTE.—The *dotted* line shows the altitude.

(C.) "Triangles upon the *same* base and between the same parallels are equal to one another" (in area).—**Euc. I., 37.** *Ex. ABC, DBC*—



NOTE.—It can be readily seen that *ABC* is a half of the parallelogram *EBCA*, and *DBC* is a half of the parallelogram *DBCF*. The parallelograms, standing on the same base *BC*, are equal, therefore the triangles are equal, as "the halves of equal things are equal."—**Euc. I., Ax. 7.**

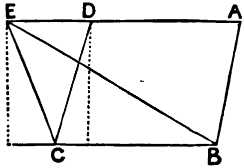
(D.) "Triangles upon *equal* bases and between the same parallels are equal to one another" (in area).—**Euc. I., 38.** *Ex. ABC, DEF*—



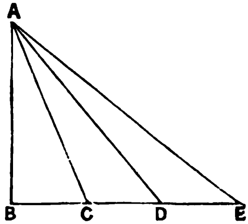
NOTE.—It can be readily seen that *ABC* is a half of the parallelogram *GBCA*, and *DEF* is a half of the parallelogram *DEFH*. The parallelograms, standing on equal bases *BC* and *EF*, are equal, therefore the triangles are equal, as "the halves of equal things are equal."—**Euc. I., Ax. 7.**

From the foregoing *notes* the truth of the following theorem will be readily seen :—

(**E.**) "If a parallelogram and a triangle be upon the *same* base and between the same parallels, the parallelogram shall be double of the triangle" (in area).—**Euc. I., 41.** *Ex. ABCD* and *EBC*—

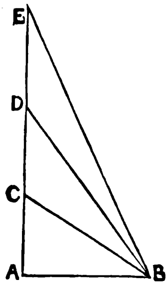


(**F.**) "Triangles of the *same* altitude are one to the other (in area) as their bases."—**Euc. VI., 1.** *Ex. ACE, ABC*—



NOTE.—The base *CE* being double of the base *BC*, the area of *ACE* is double of that of the triangle *ABC*.

(**G.**) Triangles on the *same* base have to one another the ratio that their altitudes have.—*Ex. EAB, CAB*—



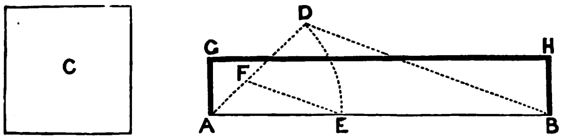
NOTE.—The altitude *EA* being triple of the altitude *CA*, the area of *EAB* is triple of that of the triangle *CAB*.

Problem 157.

To construct a **rectangle** *on a given line AB equal in area to a given* **square** *C.*

1. From *A* in *AB*, draw a straight line *AD* equal to a side of the given square, and making *any* angle with *AB*.

2. Join *BD*, and from *A*, *with radius AD*, describe an arc *DE*, meeting *AB* in *E*.



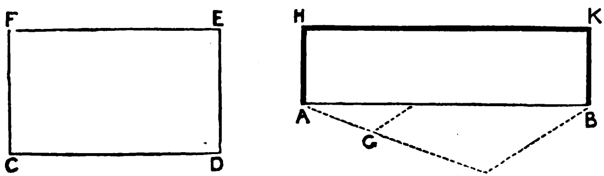
3. Through *E*, draw *EF* parallel to *BD* (**Pr. 9**). Then *AF* is a third proportional less between *AB* and a side of the square (**Pr. 143**), and *AF is equal to the second side of the required rectangle.*

4. From *A*, raise a perpendicular *AG* (**Pr. 2**) equal to *AF*, and complete the required rectangle *AGHB*, *which will be equal in area to the given square C.*

Problem 158.

To construct a **parallelogram** *on a given base AB equal in area to a given* **parallelogram** *CDEF.*

1. Find *AG* the fourth proportional less to *AB, CD, CF* (**Pr. 141**).

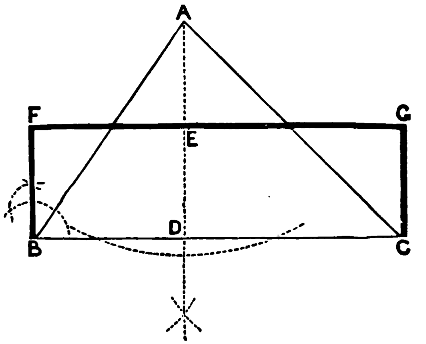


2. At *A*, raise the perpendicular *AH* equal to *AG*, and complete the parallelogram *AHKB*. *Then the parallelogram AHKB is equal to the given parallelogram CDEF.*

Problem 159.

To construct a **rectangle** *equal in area to a given* **triangle** *ABC, on one side BC of the given triangle.*

1. Draw *AD*, the altitude of the given triangle (**Pr. 21**), and bisect it in *E*. *Then ED is equal to the other side of the required rectangle, BC being one.*



2. At *B*, raise a perpendicular *BF* (**Pr. 2**) to meet the line of bisection through *E* in *F*.

3. Make *FG* equal to *BC*, and join *CG*. Then *BCGF is the required rectangle equal to the given triangle ABC, and constructed on one of its sides BC.*

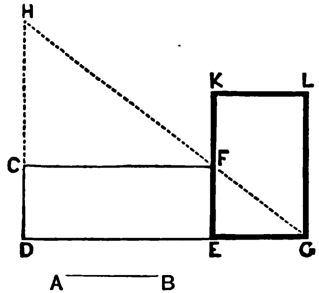
Problem 160.

To construct a **rectangle** *on a given* **line** *AB which shall be equal in area to a given* **rectangle** *CDEF.*

1. Produce the side *DE* beyond *E*, and make *EG* equal to *AB*.

2. Draw a line from *G* through *F*, to meet *DC* produced in *H*. *Then CH is equal to the other side of the rectangle, of which EG is one.*

3. From E in EF or *EF produced,* cut off EK equal to CH.

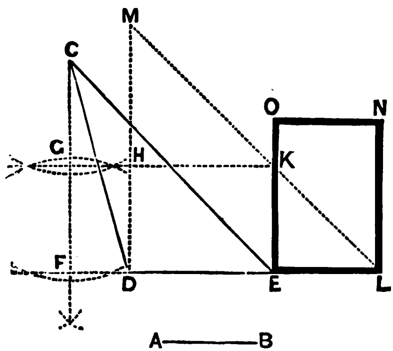


4. Make KL equal to EG, and GL equal to EK, and join KL and GL. *Then EGLK is the rectangle required, and it is constructed on EG, which is equal to the given line AB.*

Problem 161.

To construct a **rectangle** *having a given* **side** *AB, equal in area to a given* **triangle** *CDE.*

1. Find CF the altitude of the triangle CDE (**Pr. 21**).



2. Bisect CF in G (**Pr. 1**), and produce the line of bisection over DE, and erect the perpendiculars DH, EK. *Then the rectangle DEKH is equal to the triangle CDE* (**Pr. 159**).

3. Produce *DE* beyond *E*, and *DH* beyond *H;* and make *EL* equal to *AB*.

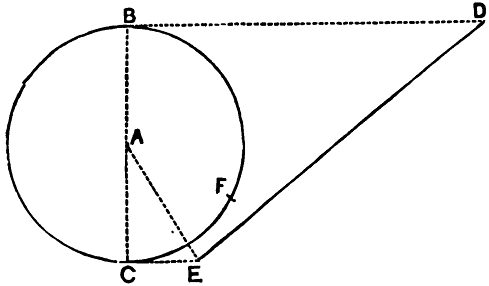
4. Draw a line from *L* through *K*, to meet *DH* produced in *M*. *Then HM is the second side of the required rectangle ELNO, and it is constructed on EL, which is equal to the given line AB.*

NOTE.—*ELNO* is equal to *DEKH* by **Problem 160**, and is therefore equal to the given triangle *CDE*.

Problem 162.

To draw a **straight line** *equal to* **half the circumference** *of a given* **circle** *A*.

1. Draw a diameter *BC*, and from *B* draw *BD* at right angles to *BC* (**Pr. 2**) and equal to *three times the radius of the circle.*



2. From *C*, draw a line *CE* at right angles to *BC*.

3. *With the radius of the circle*, cut off arc *CF* and bisect it.

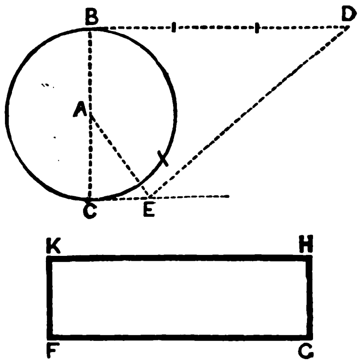
4. From the centre of the circle, draw *AE* through the point of bisection, meeting *CE* in *E*.

5. Join *ED*. *Then ED will be the required straight line equal to half the circumference of the given circle A.*

Problem 163.

To construct a **rectangle** *equal in area to a given* **circle** *A.*

1. By the last problem it will be seen that a rectangle *FGHK* can be constructed equal in area to a given



circle by making two of its sides, *FG* and *KH*, equal to the length of half its circumference as *ED*, and the other two sides *FK*, *GH*, equal to the radius *AC*.

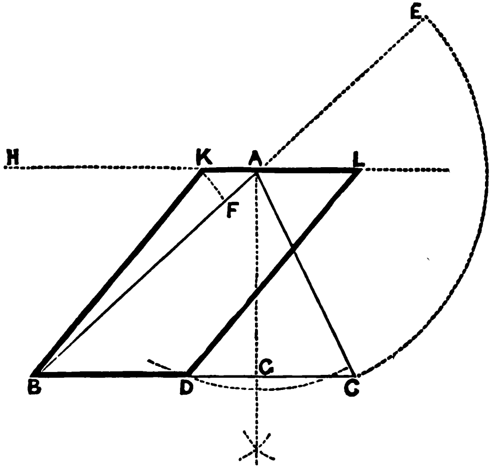
NOTE.—A *triangle* also can be constructed of the same area as a circle, by making its base equal to half the circumference of the circle, and its altitude equal to *twice* its radius (=its diameter).

Problem 164.

To construct a **parallelogram** *equal to any given* **triangle** *ABC both in* **area** *and* **perimeter.**

1. Bisect *BC* in *D* (**Pr. 1**).
2. Produce *BA* beyond *A*, and make *AE* equal to *AC*, and bisect *BE* in *F*.
3. Find the altitude *AG* of the given triangle (**Pr. 21**), and through *A* draw *AH* parallel to *BC* (**Pr. 9**).

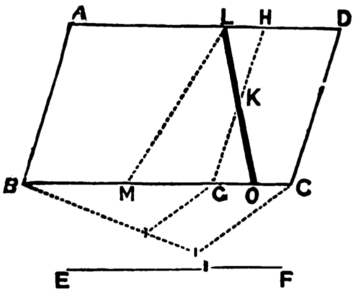
4. From *B*, *with BF as radius*, describe an arc *FK*, cutting *AH* in *K*; and from *D*, *with the same radius*, cut *HA* produced in the point *L*.



5. Join *BK*, *DL*. *Then the parallelogram BDLK is equal both in area and perimeter to the given triangle ABC.*

Problem 165.

To **divide** *any parallelogram ABCD into* **two** *parts, proportionate in area to a given divided* **line** *EF, from a* **point** *O in one side.*



1. Divide *BC into the same proportions as the given divided line EF* (**Pr. 16**) in the point *G*.

2. Through *G*, draw *GH* parallel to *CD* (**Pr.** 9), and bisect *GH* in *K* (**Pr.** 1).

3. From the given point *O*, draw a line through *K*, and produce it to meet *AD* in *L*. *Then OL will divide the given parallelogram ABCD into two parts proportionate in area to the given line EF.*

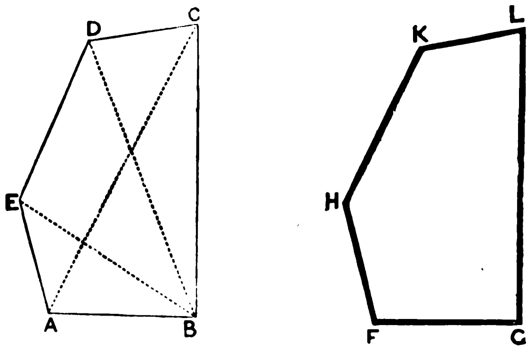
NOTE.—To divide the parallelogram into two *equal* parts from any point, say *L*, measure off *BM* equal to *DL*, and join *LM*; then *LM* will divide the parallelogram into two equal parts.

Problem 166.

To make an **irregular polygon** *equal to a given* **irregular polygon** *ABCDE.*

1. Draw a line *FG* equal to *AB*.

2. With centre *F*, *and radius AE*, describe an arc, and with centre *G*, *and radius BE*, cut it in *H*.



3. With centre *H*, *and radius ED*, describe an arc, and with centre *G*, *radius BD*, cut it in *K*.

4. With centre *K*, *and radius DC*, describe an arc, and with centre *G*, *radius BC*, cut it in *L*.

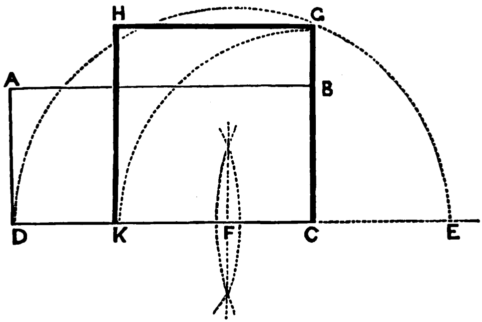
5. Join *FH*, *HK*, *KL*, and *LG*. *Then FHKLG will be the required irregular polygon.*

Problem 167.

To construct a **square** *equal in area to a given* **rectangle** *ABCD.*

1. Produce *DC indefinitely* beyond *C*, and make *CE* equal to *CB*.

2. Bisect *DE* in *F* (**Pr. 1**), and on *DE* describe a semicircle.



3. Produce the perpendicular *CB* to meet the semicircle in *G*. Then *CG* is a mean proportional between the two adjacent sides *DC*, *CB* (**Pr. 140**); and *CG is one side of the required square.*

4. On *CG*, complete the required square *CGHK* (**Pr. 34**), *which will be equal in area to the given rectangle ABCD.*

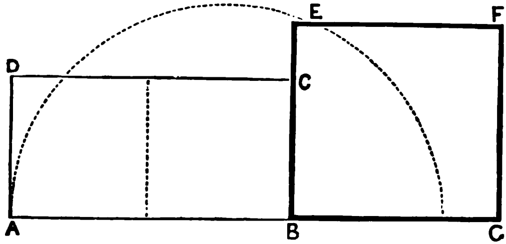
Problem 168.

To construct a **square** *that shall have an area of* **two square inches** (*or any number of square inches*).

1. Let *ABCD* be a rectangle having an area of two square inches, its side *AB* being two inches (linear), and *BC* one linear inch.

2. Find *BE* a mean proportional to the lines *AB*, *BC* (**Pr. 140**).

3. On *BE* construct a square *BEFG* equal to the rectangle

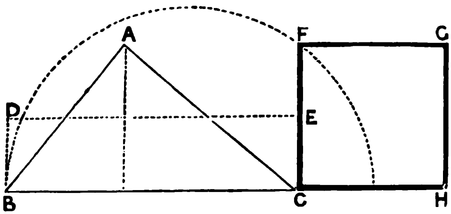


ABCD (**Pr. 167**); *then BEFG shall have an area of two square inches.*

Problem 169.

To construct a **square** *equal in area to any given* **triangle,** *ABC.*

1. Make the rectangle *DBCE* equal to the triangle *ABC* (**Pr. 159**).



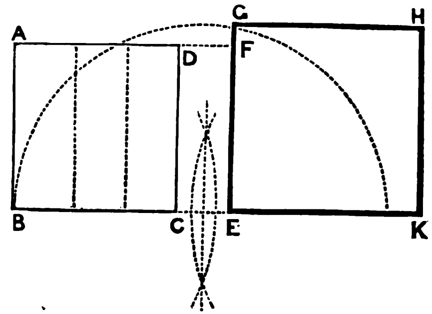
2. Find *CF* a mean proportional to the lines *BC*, *CE* (**Pr. 140**). *Then CF is a side of the required square.*
3. Complete the square *CFGH* (**Pr. 34**), *which will be equal in area to the given triangle ABC.*

Problem 170.

To construct a **square** *having an area* **one-third greater** *than that of a given* **square** *ABCD.*

1. Divide *BC* into *three* equal parts (**Pr. 15**), and *from the points of division* draw lines parallel to *BA* or *CD* (**Pr. 9**).

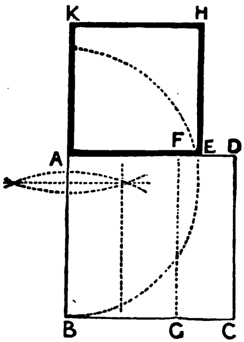
2. Produce *BC* beyond *C*, and *AD* beyond *D*, and make *CE* and *DF* each equal to one-third of *BC*.



3. Join *EF;* and find a mean proportional, *EG* (**Pr. 140**), to two adjacent sides of the rectangle *BEFA*.
4. Complete the required square of which *EG is a side* (**Pr. 34**). *Then the square EGHK is one-third greater than the given square ABCD.*

Problem 171.

To construct a **square** *having an area* **one-third less** *than that of a given* **square** *ABCD.*



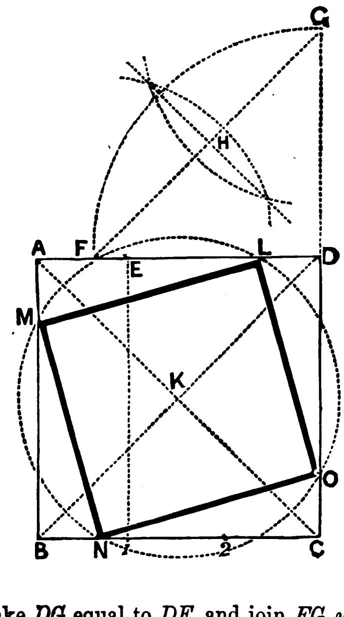
1. Divide *BC* into *three* equal parts (**Pr. 15**), and *from the points of division* draw lines parallel to *BA* or *CD* (**Pr. 9**).

2. Find a mean proportional *AE* (**Pr. 140**) to the two adjacent sides of the rectangle *BAFG*.
3. Complete the required square, of which *AE is one side* (**Pr. 34**). *Then the square AEHK is one-third less than the given square ABCD.*

Problem 172.

To inscribe within a given **square**, *ABCD, another* **square** *having its angles in the sides of the first, and being proportional in area, as* $\frac{2}{3}$.

1. Divide *BC* into *three* equal parts (**Pr. 15**), make *DE* equal to *C* 1 and join 1 *E*.
2. Find a mean proportional *DF* (**Pr. 140**), to the two sides *CD, DE,* of the rectangle 1 *CDE, which will be equal to one side of the required square.*



3. Make *DG* equal to *DF*, and join *FG, which is equal to one diagonal of the required square.*

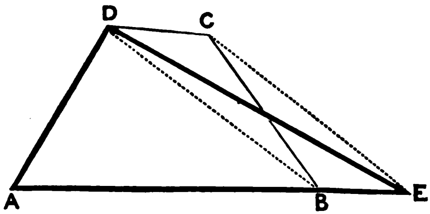
4. Bisect *FG* in *H* (**Pr. 1**), and draw the diagonals *AC*, *BD*, cutting in *K*.

5. From *K*, *with radius HF or HG*, describe a circle cutting the sides of the given square in two points each. Join the alternate points *L*, *M*, *N*, *O*, and *the required square will be inscribed within the given square ABCD.*

Problem 173.

To construct a **triangle** *equal in area to a given* **trapezium**, *ABCD.*

1. Draw a diagonal *DB*, and produce *AB indefinitely* to *E*.



2. Through *C*, draw a line *CE* parallel to *DB* (**Pr. 9**), meeting *AB* produced in *E*.

3. Join *DE*; *then ADE will be a triangle equal in area to the given trapezium ABCD.*

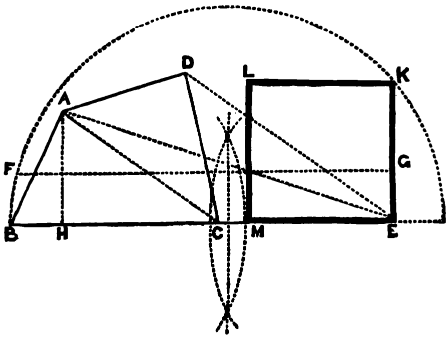
Problem 174.

To construct a **square** *equal in area to a given* **trapezium** *ABCD.*

1. Construct a triangle *BAE* equal to the given trapezium (**Pr. 173**).

2. Construct a rectangle *BFGE* on *BE* equal to the triangle *BAE*, by bisecting the altitude *AH* (**Pr. 159**), producing the line of bisection *FG*, making it equal to *BE*, and joining *BF* and *EG*.

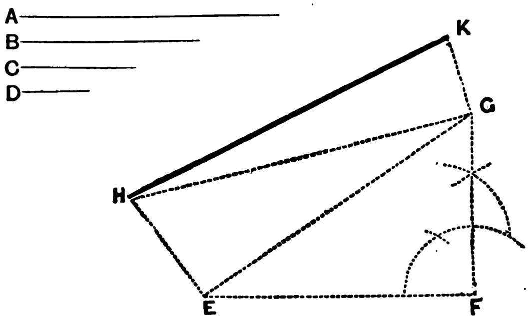
3. Construct a square equal in area to the rectangle *BFGE*, by finding a mean proportional *EK* (**Pr. 140**) to the two sides *BE*, *EG*.



4. Complete the required square *EKLM*, of which *EK is one side* (**Pr. 34**). *Then the square EKLM is equal in area to the given trapezium ABCD.*

Problem 175.

To construct a **square** *equal in area to* **any number of squares,** *of which A, B, C, D, &c., are the given sides.*



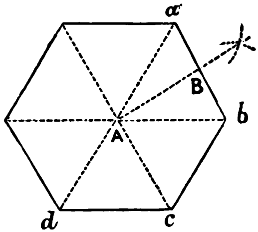
1. Place *A* and *B at right angles to each other*, *EF* being equal to *A*, and *FG* to *B*, and join *EG*; then the square constructed on the hypotenuse *EG* is equal to the squares on *A*, *B* (**Euc. I. 47**).

2. At E, one of the extremities of EG, draw EH at right angles to it (**Pr. 2**), and equal to C, and join HG; the square on the hypotenuse HG is equal to the squares on A, B, C.
3. At G, one of the extremities of HG, draw GK at right angles to HG (**Pr. 2**), and equal to D, and join HK. *Then HK will be a side of the required square, equal in area to the squares on the given sides A, B, C, D.*

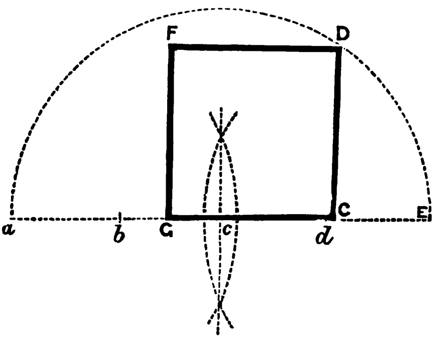
NOTE.—In this way a square may be constructed equal to *any* number of given squares.

Problem 176.

To construct a **square** *equal in area to* **any** *given* **regular polygon** *A. (Say a* **hexagon.**)



1. Resolve the given hexagon into the same number of equal triangles as the polygon has sides.



2. Draw an altitude to one of the sides as AB in the triangle Aab (**Pr. 21**).

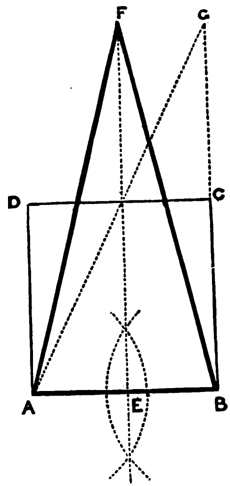
3. Find a mean proportional *CD* (**Pr. 140**) between half the perimeter of the polygon, and the altitude of one of the triangles as *CE*. *Then CD will be a side of the required square.*

4. Complete the square *CDFG* (**Pr. 34**), *which will be equal in area to the given polygon A.*

Problem 177.

To construct an **isosceles triangle,** *equal in area to, and on one side AB of a given* **square** *ABCD.*

1. Bisect *AB* in *E*, and produce the bisecting line towards *F*.



2. Produce *BC* beyond *C*, and make *CG* equal to *BC*.

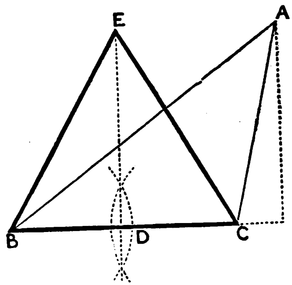
3. Join *AG*; the right-angled triangle *ABG* is equal to the given square.

4. Make *EF* equal to *BG*, and join *AF*, *BF*. *Then ABF is the required isosceles triangle.*

Problem 178.

To construct an **isosceles triangle** *equal in area to a given* **triangle** *ABC on one of its sides, BC.*

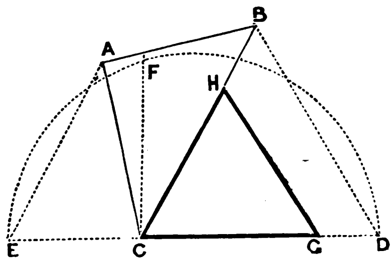
1. Bisect *BC* in *D* (**Pr. 1**), and produce the line of bisection, making *DE* equal to the altitude of the given triangle *ABC*.



2. Join *EB*, *EC*; *then the isosceles triangle EBC is equal to the given triangle ABC.*

Problem 179.

To construct an **equilateral triangle** *equal to a given* **triangle** *ABC, which is* **not equilateral.**



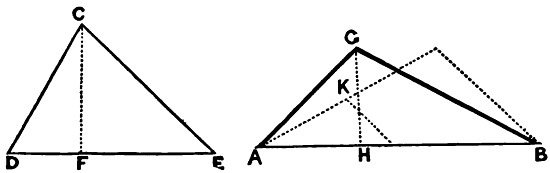
1. On *BC*, a side of the given triangle, construct an equilateral triangle *BCD* (**Pr. 18**).

2. Through *A*, draw a line *AE* parallel to *BC* (**Pr. 9**), and meeting *DC* produced in *E*.
3. Find a mean proportional to *DC* and *CE* (**Pr. 140**) thus—on *DE* describe a semicircle, and at *C* raise a perpendicular *CF* to *ED* (**Pr. 2**). Line *CF* is a mean proportional to *DC* and *CE*, *and will be the side of an equilateral triangle equal in area to the given triangle, as CGH.*

Problem 180.

To construct a **triangle** *on a given* **base** *AB, equal in area to a given* **triangle** *CDE.*

1. Let fall upon *DE* the perpendicular *CF* (**Pr. 3**). The fourth proportional less to the three lines *AB*, *DE*, *CF*, will be the perpendicular height of the required triangle.



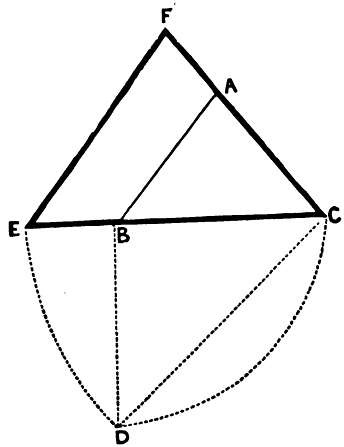
2. Find *AK* this fourth proportional less (**Pr. 141**).
3. From *any* point *H* in the base *AB* raise a perpendicular *HG equal* to *AK*.
4. Join *GA*, *GB*; *then the triangle GAB will be equal in area to the given triangle CDE.*

Problem 181.

To construct a **triangle similar** *to a given* **triangle** *ABC, but of* **twice its area.**

1. Draw *BD* at right angles to *BC* (**Pr. 2**), and equal to it.
2. Join *DC*. Then the square on *DC* is equal to the squares on *DB*, *BC* (**Euc. I. 47**), and therefore double the square on *BC*.
3. Produce *CB* to *E*, making *CE* equal to *CD*; then the square on *EC* is double the square on *BC*.

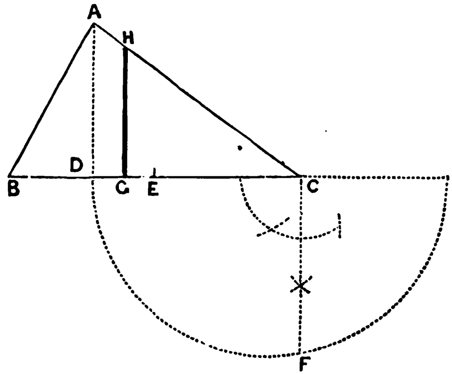
4. From E, draw EF parallel to BA (**Pr. 9**), meeting CA



produced in F. *Then the triangle FEC is similar to the given triangle ABC, and is twice its area.*

Problem 182.

To **bisect** *any* **triangle** *ABC by a* **line** *drawn* **perpendicular** *to one of its* **sides** *BC.*



1. From A drop a perpendicular AD on BC (**Pr. 3**), and bisect BC in E.

2. Find *CF* a mean proportional between *CD* and *CE* (**Pr. 140**); and make *CG* equal to *CF*.

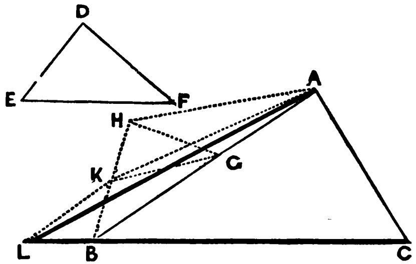
3. Through *G*, draw *GH* parallel to the altitude *AD* (**Pr. 9**), meeting *AC* in *H*. *Then GH will divide the given triangle ABC into two equal areas.*

Problem 183.

To construct a **triangle** *equal in area to any two* **dissimilar triangles** *ABC and DEF.*

1. From *B*, in *AB*, cut off *BG* equal to *EF*, and on *BG* construct a triangle *BGH* equal to the triangle *DEF*.

2. Join *AH*, and through *G* draw *GK* parallel to *AH* (**Pr. 9**), and join *AK*.



3. From *K* draw a line *KL* parallel to *AB*, cutting *CB* produced in *L*.

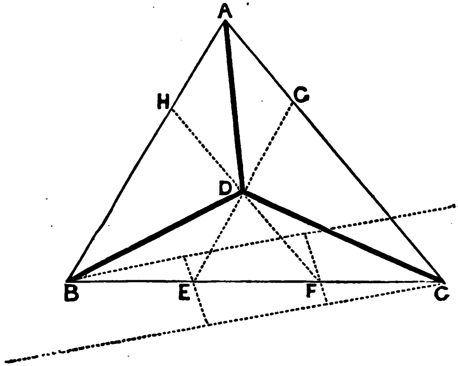
4. Join *AL*. The triangle *ALC* will then be equal to the trapezium *AKBC* (**Pr. 173**), *and equal in area to the two dissimilar triangles ABC, DEF.*

Problem 184.

To **divide** *any given* **triangle** *ABC into any number of equal parts (say in this case three), by* **lines drawn from each angle** *A, B, and C, to a* **point** *D* **within the triangle.**

1. Divide one side *BC* into *three* equal parts (**Pr. 15**) in the points *E*, *F*.

2. Through E draw EG parallel to AB (**Pr. 9**), and through F draw FH parallel to AC, cutting EG in the required point D.

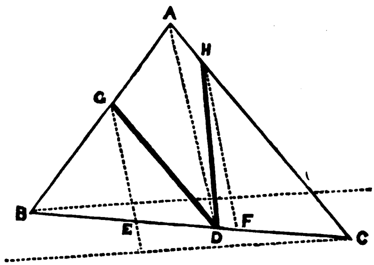


3. From each of the angles A, B, C, draw lines to the point D. *Then AD, BD, and CD will divide the given triangle ABC into the required number of parts, equal in area.*

NOTE.—The same method is to be followed for dividing the triangle into any number of *proportionate* parts.

Problem 185.

To **divide** *any given* **triangle** *ABC into any number of equal parts (say in this case three), by* **lines drawn from a given point** *D* **in one of its sides.**



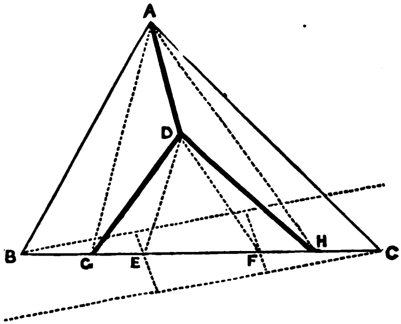
1. From the given point D draw a line to the opposite angle at A.

2. Divide the side BC, *that in which the given point is,* into *three* equal parts (**Pr. 15**) in the points E, F.
3. Through E and F draw EG and FH parallel to AD (**Pr. 9**).
4. Join DG, DH; *which lines will divide the given triangle* ABC *into the required number of parts, equal in area.*

Problem 186.

To **divide** *any given* **triangle** ABC *into any number of equal parts (say in this case three) by* **lines drawn from a given point** D **within the triangle.**

1. Divide one side BC into *three* equal parts (**Pr. 15**) in the points E, F, and join DE, DF.
2. From A, draw a line AG parallel to DE (**Pr. 9**); and



from A draw a line AH parallel to DF, and meeting BC in H.

3. Join DA, DG, DH; *which lines will divide the given triangle* ABC *into the required number of parts, equal in area.*

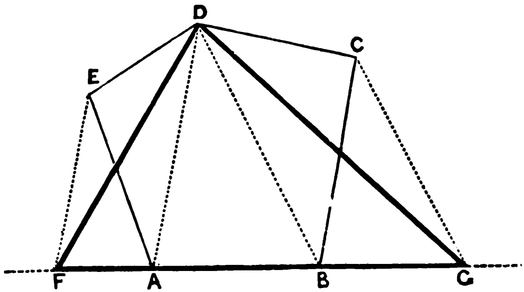
Note.—The same method is to be followed for dividing the triangle into any number of *proportionate* parts.

Problem 187.

To convert any **rectilineal figure** $ABCDE$ *into a* **triangle** *of equal area.*

1. Produce any side AB both ways *indefinitely.*

2. Join *DA*, *DB;* through *E* draw *EF* parallel to *DA*, and through *C*, draw *CG* parallel to *DB*.



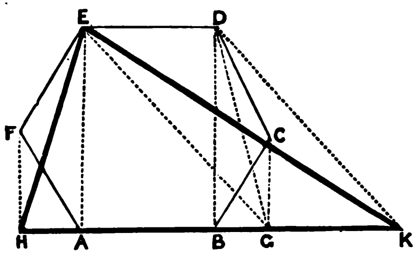
3. Join *DF*, *DG*. *Then DFG will be the required triangle, and it is equal in area to the given rectilineal figure ABCDE.*

Problem 188.

To **reduce** *any* **rectilineal figure** *to an* **equivalent** *figure having a less number of sides (e.g., a* **hexagon** *to a* **triangle**).

Let *ABCDEF* be the given hexagon.

1. Join *DB*, and through *C* draw *CG* parallel to *DB* (**Pr. 9**), cutting *AB* produced in *G*.



2. Join *DG;* then the triangle *DGB* is equal to the triangle *DCB*, and therefore *the five-sided figure AGDEF is equal to the given hexagon ABCDEF.*

3. Join *EA*, and through *F* draw *FH* parallel to *EA* (**Pr. 9**), cutting *BA* produced in *H*.

4. Join *EH;* then the triangle *EFA* is equal to the triangle *EHA*, and therefore *the quadrilateral EHGD is equal to the five-sided square AGDEF.*

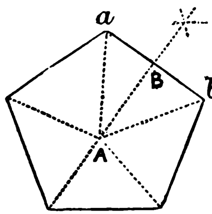
5. Join *EG*, and through *D* draw *DK* parallel to *EG* (**Pr. 9**), meeting *HG* produced in *K*.

6. Join *EK;* then the triangle *EKG* is equal to the triangle *EDG*, and therefore *the triangle EHK equals the quadrilateral EHGD, or the given hexagon ABCDEF.*

Problem 189.

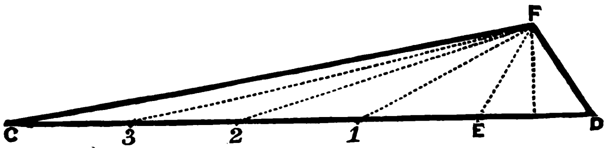
To construct a **triangle** *equal in area to* **any** *given* **regular polygon** *A* (*say a* **pentagon**).

1. Resolve the given pentagon into the same number of equal triangles as the polygon has sides.



2. Draw an altitude to one of the sides as *AB* in the triangle *Aab* (**Pr. 21**).

3. Set off the length of the five bases in one line *CD*, and on one of them, *ED*, construct a triangle *FED* equal to the triangle *Aab* (**Pr. 180**), and having the same altitude.

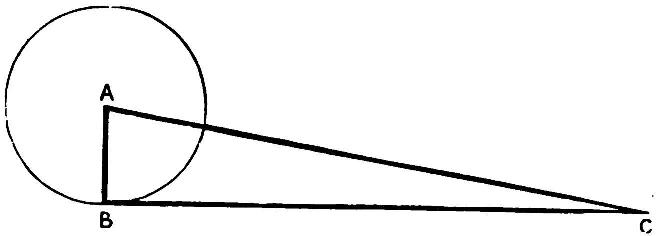


4. From the point *F* draw *F*1, *F*2, *F*3, *FC; then the triangle FCD is equal in area to the given polygon A.*

Problem 190.

To construct a **triangle** *equal in area to any given* **circle** *A.*

1. Draw *any* radius *AB*, and from *B* draw *BC* perpendicular to *AB* (**Pr. 2**), and equal in length to the circumference of the circle.

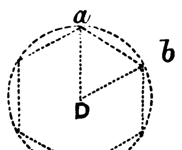


2. Join *AC*, *and the triangle ABC is equal in area to the given circle A.*

Problem 191.

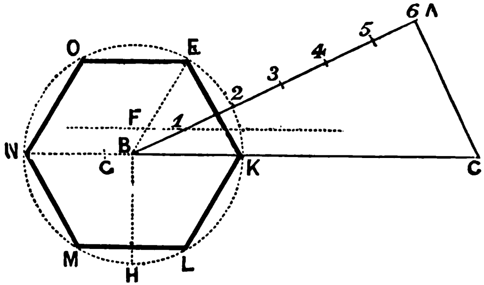
To construct **any** *regular* **polygon** (*say a* **hexagon**), *equal in area to any given* **triangle** *ABC.*

1. Divide a side *BA* of the given triangle into *as many equal parts as the required polygon has sides* (*six*) (**Pr. 15**).



2. Produce *CB indefinitely* beyond *B*, and through point 1 draw a line parallel to *BC* (**Pr. 9**).
3. Construct a hexagon *D* (**Pr. 64**), and draw lines from the centre to two of the angles, as *Da*, *Db*.
4. At *B* in *CB* make angle *CBE* equal to *aDb* (**Pr. 10**), the line *BE* cutting the parallel to *BC* in *F*.
5. On *CB* produced make *BG* equal to *BF*.

6. Find a mean proportional *BH* to the two segments *CB*, *BG* (**Pr. 140**), which is equal to the radius of the circle described from *B*.

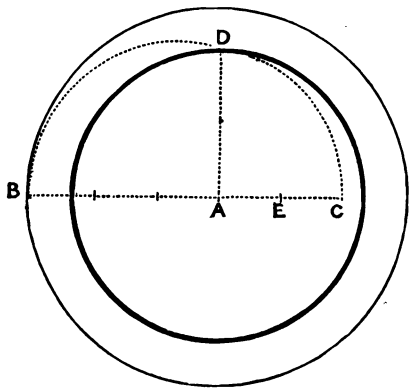


7. Within this circle *inscribe the required hexagon EKLMNO, which is equal in area to the given triangle ABC.*

Problem 192.

To construct a **circle two-thirds** *the area of a given* **circle** *A*.

1. Draw a radius *AB*, and divide it into *three* equal parts (**Pr. 15**).



2. Produce it to *C*, making *AC* equal to *two* of the equal parts.

3. Find AD the mean proportional between AB and AC (**Pr. 140**), *which is the radius of the required circle.*

4. From centre A, *with radius* AD, *describe the inner circle, whose area is two-thirds of that of the given circle* A.

NOTE.—In the same manner, if the required circle is to be *one*-third of the given circle, mark off AE equal to *one*-third of AB. Then find the mean proportional between AB and AE, which will be the radius of the required circle.

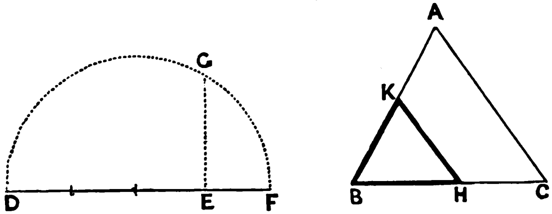
Problem 193.

To construct **any rectilineal figure,** *whose area shall have a* **given proportion** *to any other* **rectilineal figure** *of the same kind* (*say* **one-third**).

(**A**) *To construct a* **triangle** *one-third of a given triangle* ABC.

1. Draw DE equal to the side BC. As *one-third* the area is required, produce DE, making EF equal to *one-third* of DE.

2. Find EG a mean proportional to DE, EF (**Pr. 140**). *Then* EG *is equal to a side of the required triangle.*



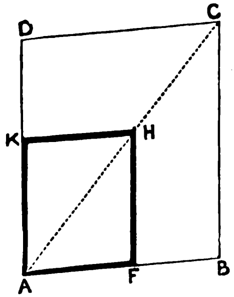
3. Make BH equal to EG, and draw HK parallel to AC. *Then* BHK *will be the required triangle, and it is one-third of the given triangle* ABC.

(**B**) *To construct a* **parallelogram** *one-third of a given parallelogram* $ABCD$.

1. As in case **A**, find EG the mean proportional (**Pr. 140**).

2. On AB, mark off AF equal to EG, and join AC.

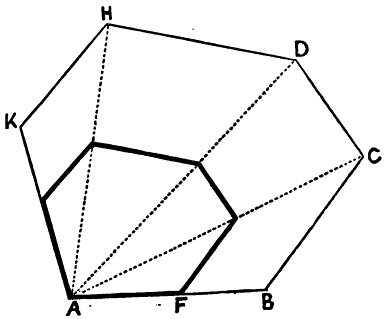
3. Draw *FH*, *HK*, parallel to *BC*, *CD*. *Then AFHK will*



be the required parallelogram, and it is one-third of the given parallelogram ABCD.

(C) *To construct an* **irregular rectilineal figure** *one-third of a given irregular rectilineal figure ABCDHK.*

1. As in case **A**, find *EG* the mean proportional (**Pr. 140**).
2. On *AB*, mark off *AF* equal to *EG*, and from *A*, draw *AC*, *AD*, *AH*.



3. From *F*, commence drawing a series of lines parallel to the sides of the given figure, *and the smaller rectilineal figure will be constructed.*

NOTE.—As in the case of the circle, if the figures required be *any* other proportion of the given figures, *e.g.*, *three-fifths ; make EF three-fifths of DE*, and find the mean proportional as before. *That* will be equal to a side of the figure required.

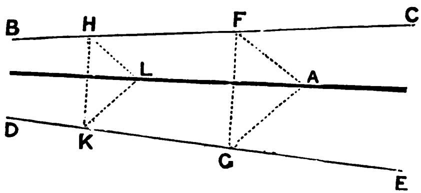
Section XII.

MISCELLANEOUS PROBLEMS.

Problem 194.

Through a given **point** *A, to draw a* **line** *which would, if produced,* **pass through the angular point** *towards which the two given* **lines** *BC, DE* **converge.**

1. Draw *any convenient* line *FG*, and join *FA*, *GA*.
2. Draw *any* line *HK* parallel to *FG* (**Pr.** 9).



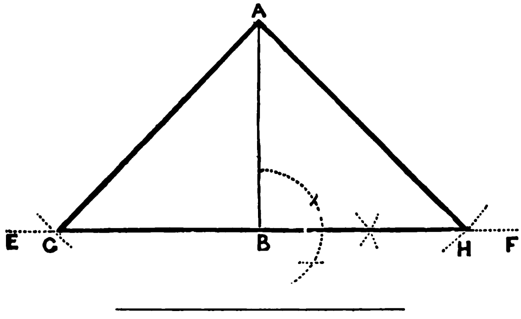
3. Through *H* and *K*, draw *HL* and *KL* parallel to *FA* and *GA* (**Pr.** 9), meeting each other in *L*.
4. Through *A* and *L*, draw *AL*, *which produced is the convergent line required.*

Problem 195.

To construct an **isosceles triangle,** *having given its* **altitude** *AB, and CD the* **length of its equal sides.**

1. Through *B*, draw *EF* of *unlimited* length, and at right angles to *AB* (**Pr.** 2).

2. From centre *A*, *with the distance CD,* cut *EF* in *G* and *H*.

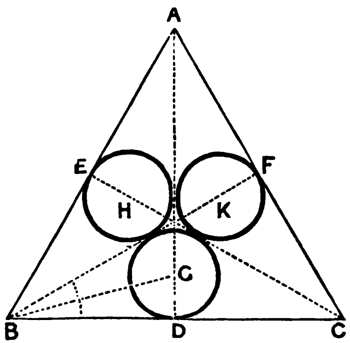


3. Join *AG*, *AH*. *Then AGH will be the required isosceles triangle, having the altitude AB and each of its equal sides equal to the given line CD.*

Problem 196.

To inscribe **three circles** *in a given* **equilateral triangle** *ABC, each touching the other two, and* **one side of the triangle.**

1. Bisect each side of the triangle (**Pr. 1**), and draw the lines *DA*, *EC*, *FB*.

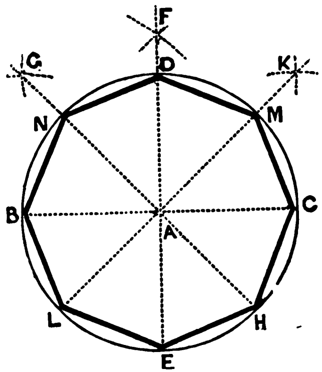


2. Bisect the angle *FBC*, by a line cutting *AD* in *G* (**Pr. 4**).
3. From *E* and *F*, cut off *EH* and *FK* equal to *GD*; *then G, H, K, are the centres of the required circles, and GD the radius.*

Problem 197.

To construct a regular **octagon** *within a given* **circle** *A, making* **one of its angles** *coincide with a given* **point** *B.*

1. From *B*, draw a diameter *BC*.



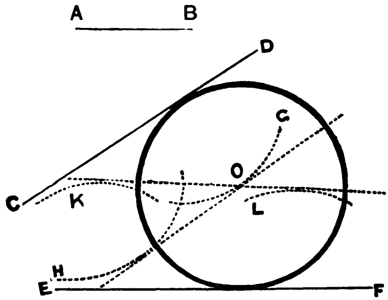
2. Draw the diameter *DE* at right angles to *BC*, *by using B and C as centres*, and describing arcs in *F*, and joining *FA*.
3. Bisect angle *DAB* (**Pr.** 4) by *GH*, also bisect angle *DAC* by line *KL*.
4. Join *BN*, *ND*, *DM*, *&c.*, *and the required octagon will be inscribed within the given circle A.*

Problem 198.

To describe a **circle** *of a given* **radius** *AB which shall be* **tangential** *to any two given* **converging lines** *CD, EF.*

1. With *any* two points on *CD*, as centres, *and radius AB*, describe arcs *G* and *H*.
2. With *any* two points on *EF*, as centres, *and radius AB*, describe arcs *K* and *L*.

3. Draw lines tangential to each pair of arcs (**Pr. 56**); and *O*, their point of intersection, *is the centre of the required circle.*

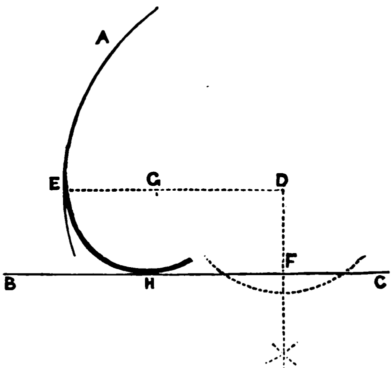


4. With centre *O*, *and radius AB, describe the required circle, which shall be tangential to the given converging lines CD, EF.*

Problem 199.

To describe an **arc,** *which shall be* **tangential** *to a given arc A, and a given* **line** *BC.*

1. *D* is the centre of the given arc *A*.



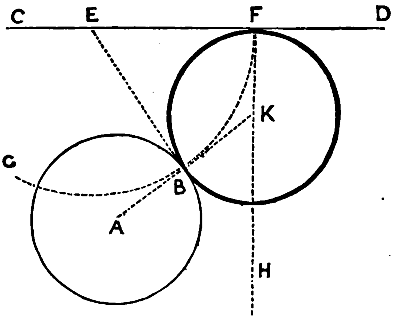
2. Through *D*, draw *DE* parallel to *BC* (**Pr. 9**).
3. From *D*, draw *DF* perpendicular to *BC* (**Pr. 2**).

4. On *ED*, mark off *EG* equal to *DF*.
5. With centre *G*, *and radius GE, describe the required arc EH, which will be tangential to the given arc A, and given line BC.*

Problem 200.

To draw a **circle,** *which shall touch a given* **circle** *A in a given* **point** *B, and also a given* **line** *CD.*

1. Find *A*, the centre of the given circle (**Pr. 45**), and join *AB*.
2. Draw the tangent *BE* (**Pr. 54**), meeting *CD* in *E*.



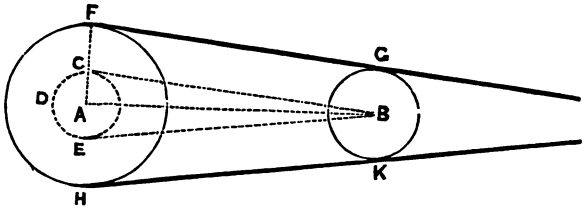
3. From *E* as centre, *with radius EB*, describe the arc *FBG*, meeting *CD* in *F*.
4. From *F*, draw *FH* perpendicular to *CD* (**Pr. 2**), and produce *AB* to meet the perpendicular in *K*; *then K is the centre of the required circle, of which KF is the radius.*

Problem 201.

To draw one or two **exterior tangents** *common to two given* **circles** *A and B.*

1. Find the centres of the given circles *A* and *B* (**Pr. 45**), and join them.
2. From the centre *A*, *with a radius equal to the difference between the radii of the given circles*, describe the circle *CDE*.

3. From the centre B, draw BC a tangent to the circle CDE (**Pr. 55**).
4. Join AC, and produce it to cut the given circle in F.

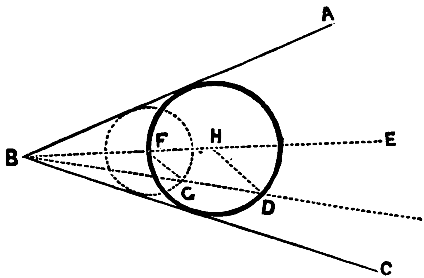


5. Through F, draw FG parallel to CB (**Pr. 9**), *then FG is a tangent common to the given circles A and B. In the same manner, HK may be drawn, another tangent common to the given circles A and B.*

Problem 202.

To inscribe a **circle** *in a given* **angle** *ABC, which shall pass through a given* **point** *D.*

1. Bisect the angle ABC by the line BE (**Pr. 4**).
2. In BE, take any *convenient* point F, and *with F as centre*, describe a circle, touching the lines AB, BC.

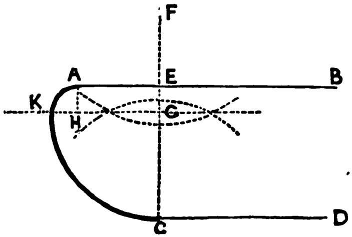


3. Join BD, cutting the circle in G, and draw the radius FG.
4. Through D, draw DH parallel to FG (**Pr. 9**), *then H is the centre of the required circle.*
5. From centre H, *with radius DH, inscribe the required circle in the given angle ABC.*

Problem 203.

To **join the extremities** *of any two given* **parallel lines** *AB*, *CD*, *by a* **pair of arcs,** *which shall touch each other, and the ends of the lines tangentially.*

1. At *C*, erect a perpendicular *CE* (**Pr.** 2), and produce it to *F*, making *EF* equal to *EA*.



2. Bisect *CF* in *G* (**Pr.** 1), and through *G*, draw a line parallel to *AB* (**Pr.** 9) of *indefinite length towards the left.*

3. Mark off *GH* equal to *AE*. With centre *H*, *and radius HA*, describe the arc *AK*.

4. With centre *G*, *and radius GK*, describe the arc *KC*. *Then the given parallel lines AB*, *CD*, *will be joined by the required pair of arcs AK*, *KC*.

Problem 204.

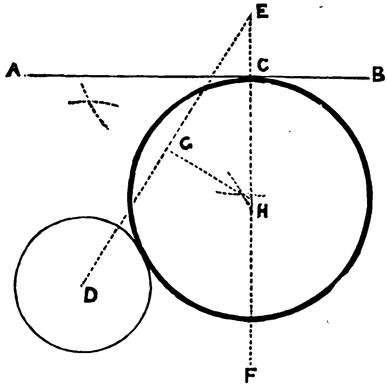
To construct a **circle** *which shall touch a given* **line** *AB in the given* **point** *C*, *and also a given* **smaller circle** *D*.

1. Through the point *C*, draw a perpendicular *EF* (**Pr.** 2) of *unlimited* length.

2. Find the radius of the given circle *D*, and from *EF* cut off *CE* equal to it, and join *DE*.

3. Bisect *DE* in the point *G* (**Pr.** 1), and draw *GH* perpendicular to it, meeting *EF* in *H*; *then H is the centre of the required circle.*

4. From *H, with radius HC,* describe the required circle

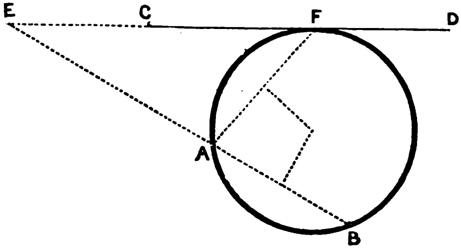


which will touch the given line AB in the given point C, and also the given smaller circle D.

Problem 205.

To construct a **circle** *which shall pass through* **two** *given* **points** *A and B, and shall touch a given* **line** *CD.*

1. Draw the straight line *BA*, and produce it to meet *DC* produced in *E.*
2. Find a mean proportional between the lines *BE* and *EA* (**Pr. 140**), and from *E* on the line *ED*, mark off *EF*



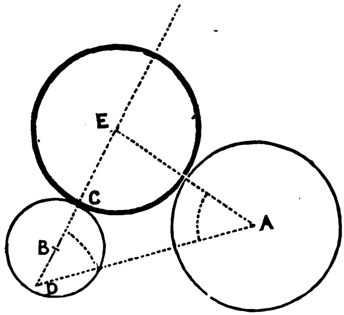
equal to the mean proportional ; *then F is the point in the given line CD, through which the required circle will touch it.*

3. Through the three points *B, A, F*, *describe the required circle BAF* (**Pr. 46**).

Problem 206.

To draw a **circle** *externally* **tangential** *to two given* **unequal circles** *A and B, and touching one of them in a given* **point** *C.*

1. Find the centres of the given circles *A* and *B* (**Pr. 45**).
2. Join *BC;* and produce it *indefinitely.*
3. On *BC* produced, mark off *CD equal to the radius of the larger circle A*, and join *AD.*



4. From *A*, draw *AE* to meet *DC* produced in *E*, and making with *DA* an angle equal to *ADE* (**Pr. 10**); *then E is the centre of the required circle.*
5. From centre *E*, *with radius EC*, describe the required circle, *which will be tangential to the two given circles A and B.*

Problem 207.

To **change** *any given* **rectilineal figure** *ABC into* **another** *rectilineal figure of* **equal area,** *but having* **one side more, &c.**

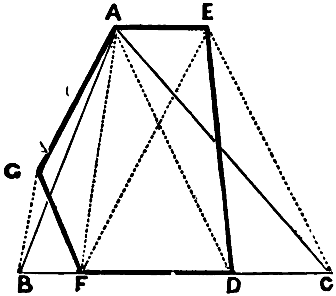
Let the given figure *ABC* be a triangle.

1. Assume a point, *D*, as one of the angles of the *four*-sided figure to be obtained, and join *AD.*
2. Draw a line from *A* in the same direction as *DC* and parallel to it (**Pr. 9**).

3. Draw a line from *C* parallel to *AD*, and meeting *AE* in *E*; and join *DE*. *Then the* **four**-*sided figure BAED will be equal in area to the given triangle ABC.*

Next,

1. Assume a point, *F*, as one of the angles of the *five*-sided figure to be obtained, and join *AF*.



2. Draw a line from *B* parallel to *AF* (**Pr.** 9), and join *EF*.
3. Draw a line from *A* parallel to *EF*, and meeting *BG* in *G*, and join *GF*. *Then the* **five**-*sided figure FGAED will be also equal in area to the triangle ABC.*

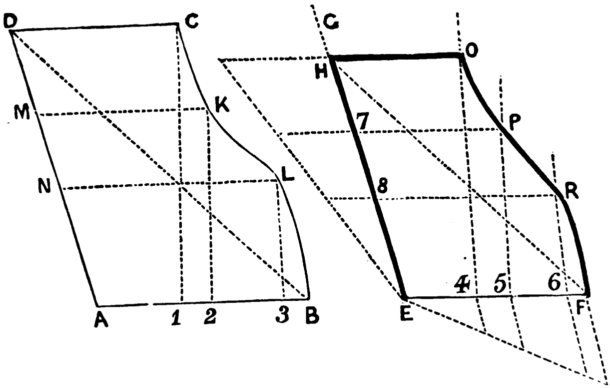
NOTE.—In the same manner, a figure of *six*, *seven*, &c., sides may be obtained, by assuming a *new* angular point in each case.

Problem 208.

To make a **reduced copy** *of any given* **figure** *ABCD, making the given* **line** *EF correspond to AB.*

1. Make angle *FEG* equal to *BAD* (**Pr.** 10), and angle *EFG* equal to *ABD*. Then *EH* will have a correct *proportion* to *AD*.
2. On the curve *CB*, mark *any* number of points, say two *K*, *L*.
3. From *C*, *K*, *L*, drop perpendiculars on *AB* (**Pr.** 3) to points 1, 2, 3.
4. Draw *KM*, *LN*, parallel to *AB* (**Pr.** 9).
5. Divide *EF*, *EH*, *proportionally* to the divisions on *AB*, *AD* (**Pr.** 16), as shown by the dotted lines in the figure.

6. From 4, 5, 6, erect perpendiculars (**Pr. 2**), and draw *HO*, 7*P*, 8*R*, parallel to *EF* (**Pr. 9**). *Then HO corresponds to DC, and points P and R to points K and L.*



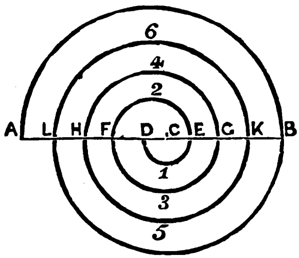
7. Draw the curve *OPRF*, and *EFOH will be a reduced copy of the figure ABCD.*

NOTE.—In a similar manner, an *enlarged* copy of any given figure may be made.

Problem 209.

To construct a **common spiral,*** *the given diameter being AB.*

1. Take *any* point *C* in *AB* as the *eye* of the required spiral.



* A spiral is a curve which, as it revolves once or more round some fixed point called its centre, recedes regularly from that centre.

M

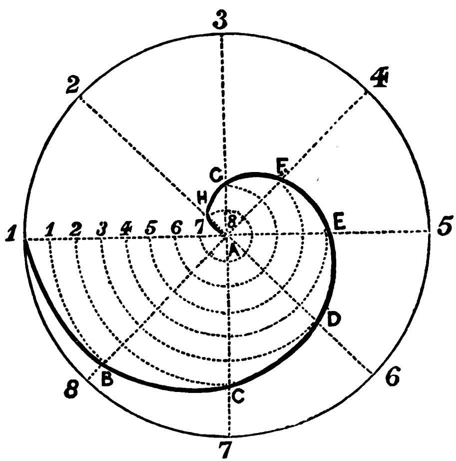
2. From *C*, as centre, ***with any radius***, describe the semicircle *D*1*E*.
3. From *D* as centre, ***with radius DE***, describe the semicircle *E*2*F*.
4. From *C* as centre, ***with radius CF***, describe the semicircle *F*3*G*.
5. From *D* as centre, ***with radius DG***, describe the semicircle *G*4*H*, &c., &c.

Note.—In this manner, a common spiral may consist of *any* number of semicircles, the points *C* and *D* being *alternately* the centres of the required semicircles.

Problem 210.

To construct a **spiral of one revolution.**

1. Divide the given circle *A* into *any* number of equal parts (say in this case *eight*), and draw radii to each point of division 1, 2, 3, &c.



2. Divide *one* of the radii, say *A*1, into the *same* number of equal parts (**Pr. 15**), and number them from the circumference 1, 2, 3, &c.
3. From *A*, ***with radii*** *A*1, *A*2, *A*3, *&c.*, describe arcs on *A*1,

cutting the corresponding radii 8, 7, 6, &c., in *B*, *C*, *D*, &c.

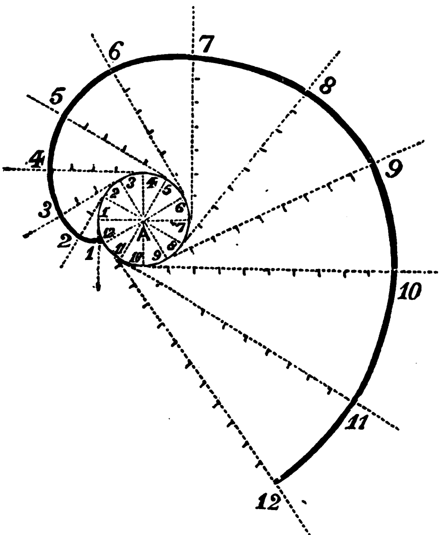
4. Through points *B*, *C*, *D*, &c., *draw the required spiral* 1*BCD*.....*A*.

NOTE.—The above spiral is usually termed the Archimedes spiral of one revolution, in honour of Archimedes, one of the most celebrated mathematicians of antiquity, who flourished about 300 A.C.

Problem 211.

To construct the **involute** *of a given* **circle** *A*.

1. Divide the given circle *A* into *any* number of equal parts (say in this case *twelve*), and draw radii to each point of division 1, 2, 3, &c.



2. From the points of division 1, 2, 3, &c., draw tangents (**Pr. 54**), *all being produced in the same direction.*

3. On the tangent drawn from point 1, mark off a space equal to *one*-twelfth of the circumference.

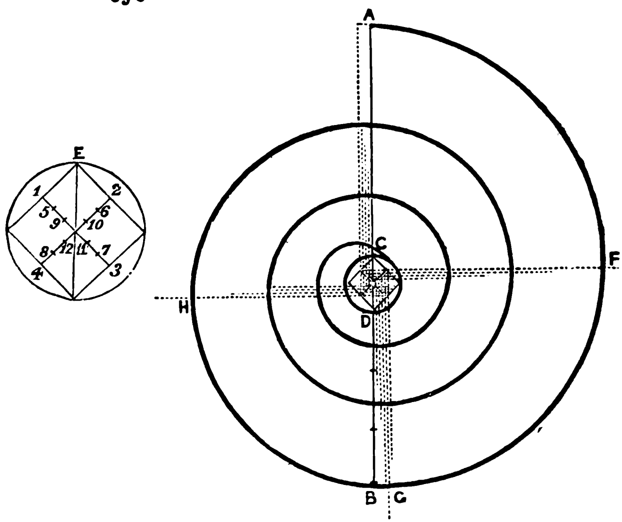
4. On the tangents drawn from points 2, 3, 4, &c., mark off spaces equal to *two, three, four*-twelfths, &c., of the circumference; *the tangent thus drawn from point* 12 *will be equal to the circumference of the circle.*

5. Through the outer extremities of the several tangents *draw the required involute.*

Problem 212.

To construct the **spiral,** *known as the* **Ionic volute,** *its longest* **diameter** *AB being given.*

1. Bisect *AB* in *C* (**Pr. 1**).
2. Divide *BC* into *four* equal parts (**Pr. 15**), and let *CD* be one of those parts.
3. On *CD* as *diameter*, describe a circle, which is called the **eye** of the volute.



4. In this circle, inscribe a square, having two **vertical** diameters.
5. Divide each of these diameters into *six* equal parts, and number the divisions, as shown in *E, the eye enlarged.*

6. Produce 1, 2, *indefinitely* beyond 2, and from centre 1, *with radius* 1*A*, describe the arc *AF*.

7. Produce 2, 3, *indefinitely* beyond 3, and from centre 2, *with radius* 2*F*, describe the arc *FG*.

8. Produce 3, 4, *indefinitely* beyond 4, and from centre 3, *with radius* 3*G*, describe the arc *GH*.

9. By proceeding in this manner, the required volute is completed at point *C*. *The radius for each successive arc is obtained by producing a line from the preceding centre through the point next in advance.*

Section XIII.—SCALES.

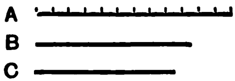
In geometrical drawing, it is often required to make a copy of an object much smaller than the object itself. For this purpose, we must make use of a *scale*, so that the several portions of the object may be drawn *proportionally*.

Scales are of various kinds; *e.g.*, *plain*, *diagonal*, and a *scale of chords*. The simplest form of scale is the **plain scale**, which consists of a line divided into equal (or unequal) portions of various lengths, each portion representing some fixed measurement. For example, let the given line *AB*, which is in reality about 3 in. in

length, represent an *actual* length of 3 yards; then one-third of the given line; *i.e.*, *AC*, will represent 1 yard; one-third of *AC* will represent 1 foot, &c. &c.

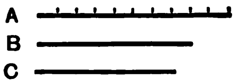
Note.—Such a scale is called a scale of $\frac{1}{36}$, because the whole line *AB* is $\frac{1}{36}$ of the distance which it represents, *i.e.* 3 in. $= \frac{1}{36}$ of 3 yards, or 108 inches. In this case, the fraction ($\frac{1}{36}$) is called the *representative* fraction of the scale.

Moreover, we may make a line of *any* length correspond to a foot, *e.g.*, 1 in. as in *A*, $\frac{7}{8}$ in. as in *B*, $\frac{3}{4}$ in. as in *C*.



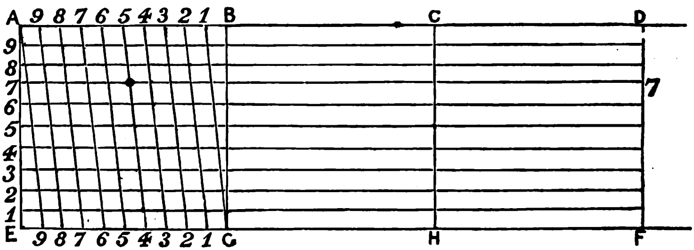
By dividing each of these lines into *twelve* equal parts, each part will correspond to an inch. Such scales are called "*duodecimal* scales."

Note.—Sometimes the line corresponding to a foot, as in the preceding, is divided into *ten* equal portions, *e.g.*—



Such scales are called "*decimal* scales."

Diagonal Scale.—A diagonal scale is a scale used for measuring more minute distances than can be done by an ordinary plain scale. It is usually divided into 100ths.



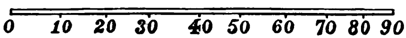
Its construction is as follows:—Any indefinite straight line is taken, from which a distance *AB* is set off according to the intended length of the scale; repeat *AB* any number of times as *BC*, *CD*, &c. Draw *EF* parallel to *AD* at any convenient distance from it; and draw the perpendiculars *AE*, *BG*, *CH*, &c. Divide *AB* and *AE* each into *ten* equal parts, and through 1, 2, 3, &c., draw lines parallel to *AD;* and through 1, 2, &c. (on the line *AB*), draw 1*G*, 2 1, 3 2, &c., as in the above figure.

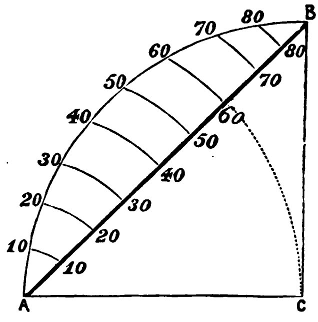
Now whatever number *EG* represents, *G*1 will be the tenth of it, and the subdivisions in the vertical direction *GB* will be each 100th part. For example, if *EG* be a unit, the small divisions in *EG*, viz. *G*1, 1 2, &c., will be 10ths, and the divisions in the altitude will be the 100th parts of a unit.

To take any number off the scale, say $2\frac{47}{100}$, *i.e.* 2·47; place one foot of the compasses at *F*, and extend the other to the division marked 4 (on *GE*); then move the compasses upward, keeping one foot on the line *FD*, and the other on the line 4 5; till the seventh interval is reached, and the extent on the compasses will be that required.

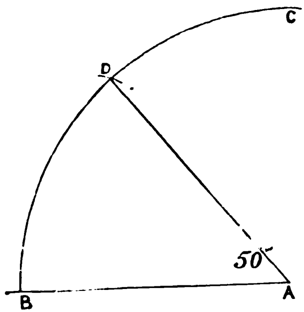
Scale of Chords.—A scale of chords is a scale by means of which, instead of a protractor or geometrical construction, angles of any number of degrees can be measured or constructed. When marked on a flat protractor, it is usually indicated by the sign *C*, or *CHO.*

Thus *CHO*





Its construction is as follows:—Any quadrant *ABC* is taken, and its arc *AB* is divided into 9 equal parts of 10° each; thus, *with the radius of the arc as radius*, points 30° and 60° are marked off on the arc *AB*. Then each portion is divided into three equal parts by trial, and each point of division is numbered in tens of degrees from 0 to 90. The chord of the arc *AB* is then drawn, and from *A* as centre, with the points of division as radii in succession, arcs are described cutting the chord *AB* in points numbered similarly to the arc; thus transferring the degrees in the arc to a straight line, from either of which the same measurements may be taken.



Thus, at the point A, in AB to make *any* angle with AB, say 50°, we take *the distance from* 0 *to* 60° *as radius*, and from A as centre with AB as radius, we describe the arc BC.

We then take the distance from 0 to 50°, and mark it off from B to D. Draw DA, then angle $DAB = 50°$.

NOTE 1.—*Under all circumstances*, describe the arc BC with the distance from 0 to 60° as radius.

NOTE 2.—In the scale of chords, the divisions diminish from 0 to 90.

END OF PLANE GEOMETRY.

PRACTICAL SOLID GEOMETRY.

A COURSE OF PROBLEMS

IN

PRACTICAL SOLID GEOMETRY.

Section I.—DEFINITIONS, &c.

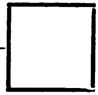
1. The preceding portion of this work consists of drawing *plane* figures. We now come to the consideration of drawing solid objects *geometrically.* Hitherto the various figures drawn have had only length and breadth, but a solid object has *another* dimension, viz., thickness or solidity.

2. It must here be noted that a solid may be represented in two distinct ways, viz., *perspectively* and *geometrically.* When an object is drawn perspectively, it is drawn as it *appears* to one from any given point of view; but when it is drawn geometrically, it is drawn as it *actually* is, its true proportions and size being represented according to scale.

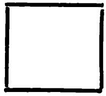
3. It follows that, in drawing a solid object geometrically, *three* dimensions have to be delineated upon a plane surface. To this end, we make *two distinct drawings,* one which represents the exact space it covers, as it would be seen when looked at *from above,* and another which represents its vertical appearance, as it would be seen when looked at *in front.* The *former* of these is called the **plan,** and shows the length and breadth; the *latter* is termed the **elevation,** and shows the length and height.

4. From a consideration of the following illustrations, it will be more readily seen what is understood by plan and elevation, *e.g.*, First,

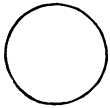
(A) The **plan** of a cube is represented thus—



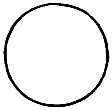
(B) ,, of a *rectangular* prism thus—



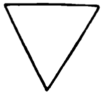
(C) ,, of a cone thus—



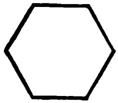
(D) ,, of a cylinder thus—



(E) ,, of a *triangular* prism thus—

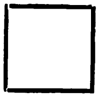


(F) ,, of a *hexagonal* prism thus—

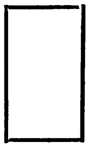


Secondly,

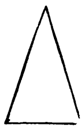
(A) The **elevation** of a cube is represented thus—



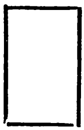
(B) The **elevation** of a *rectangular* prism thus—



(C) ,, of a cone thus—

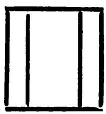


(D) ,, of a cylinder thus—



(E) ,, of a *triangular* prism thus—

(F) ,, of a *hexagonal* prism thus—

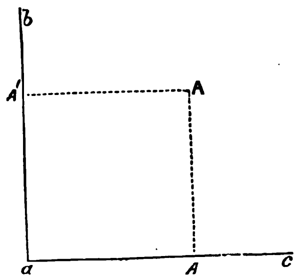


5. To any one not conversant with the principles of solid geometry, the above drawings convey no idea of a cube, cone, &c., both plan and elevation being represented as a surface, drawn on the same plane—whereas they really represent objects as *covering two planes at right angles to one another.*

6. These two "**planes of projection,**" as they are called, are distinguished as the *horizontal* plane and the *vertical* plane. They might be conveniently illustrated by the floor and walls of a room; the floor representing the horizontal plane, and the several walls so many vertical planes. The line in which the floor and any given wall intersect each other is called the "**line of intersection**;" and sometimes the ground line, or base line.

7. The two drawings which represent the plan and elevation of an object are in solid geometry called the projections of that object. Now as every solid is bounded by planes or surfaces, surfaces by lines, and lines by points, we proceed to show what is meant by the projection of a *point*, and of a *line* on the two planes of projection.

8. **First,** the projection of a *point* is obtained thus—



Let *bac* be the end view of a sheet of paper, folded in such a manner as to form a right angle at *a*. Then *ba* may be regarded as an end view of the vertical plane of projection, and *ac* an end view of the horizontal plane of projection.

Let A be a point in space. It is required to find its projections upon *ab*, *ac*.

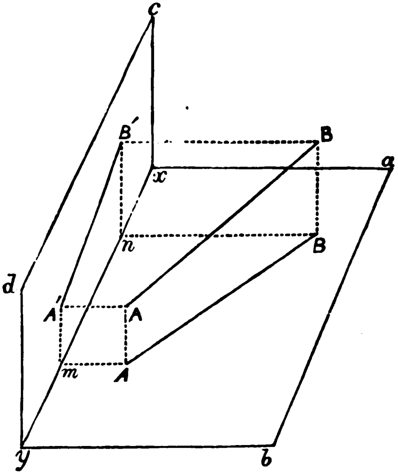
From A, draw A*A′* perpendicular to *ba*, and A*A* perpendicular to *ac*. Then the points *A*, *A′*, where the perpendiculars meet the given planes, are the projections of the point A in space.

Note 1.—The projection of a point upon a plane is the foot of a perpendicular let fall from the point upon the given plane.

Note 2.—The line which projects a point upon a plane, is termed the *projector* of that point, *e.g.*, A*A′*, A*A* are the projectors of the given point A.

Note 3.—From this it is evident that, when the projections of a point are given, the point may be found, since it is the point of intersection of the projectors of the point.

9. Secondly, the projections of a *line* are obtained thus—



Let *abxy* be the horizontal plane of projection, and *cdyx* the vertical, also let AB be the position of a line in space.

It is required to find the projections of the line AB upon *abxy* and *cdyx*.

The projection of B upon the plane *abxy* is the foot of the perpendicular let fall from B upon the given plane, say point *B*. Similarly, the projection of A upon the plane *abxy* is the foot of the perpendicular let fall from A, say point *A*.

Join *AB;* then *AB* is the plan or projection of AB upon the *horizontal* plane.

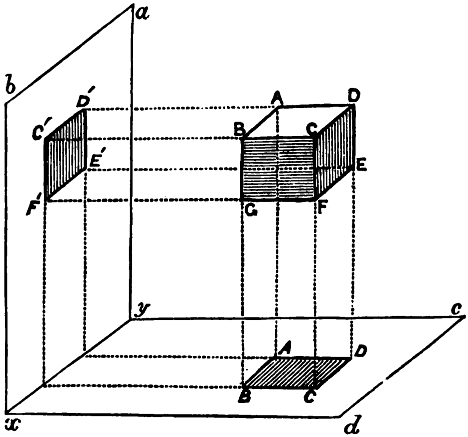
Next, the projection of B upon the plane *cdyx* is the foot of the perpendicular drawn from B to the given plane, say point *B′*. Similarly, we obtain the projection of A upon the given plane, say *A′*.

Join *A′B′;* then *A′B′* is the elevation or projection of AB upon the *vertical* plane.

Note 1.—Having found *A*, *B*, the projections of A, B, upon the horizontal plane, the elevation of *A′B′* is thus found—*Am*, *Bn* are drawn at right angles to the plane *cdyx*, meeting *xy*, the intersecting line of the two planes, in *m* and *n*. From *m* and *n* lines are drawn parallel to A*A*, and B*B;* then the intersections *A′* and *B′* of these lines, with the perpendicu- [illegible] let fall from A and B, will be the required projections.

NOTE 2.—From the elevation *A'B'*, it may be readily seen how we obtain the plan *AB*—the operation being just the converse of that shown in the preceding note.

10. As soon as the foregoing projections are thoroughly understood, the student will easily comprehend the projection of a *solid;* *e. g.*—



Let *abxy* and *cdxy* be the planes of projection, and ABCD, &c., the position in space of a regular solid. It is required to find the projection of the solid upon the two given planes.

The plan of C will be the foot of a perpendicular let fall from C upon the horizontal plane *cdxy*. Let *C* be its plan. In the same manner we find *B*. Join *BC*; then *BC* is the plan of the line BC. In the same manner we find *AD*, the plan of the line AD. Join *AB* and *CD;* then *ABCD* is the plan or projection of the given solid ABCD, &c., upon the *horizontal* plane of projection.

The intersections of the perpendiculars from the points C, D, E, F, with the plane *abxy* will give the elevation, or projection of the solid upon the *vertical* plane. The plane B*BC*C, passing through the line BC, projects that line upon *cdxy*. Also, the plane C*C'D'*D passing through the line CD, projects that line upon *abxy*.

11. The line BC, and all lines parallel to it, are parallel to the *horizontal* plane of projection ; and the line CF, and all lines parallel to it, are parallel to the *vertical* plane of projection. Also, the line BC, and all lines parallel to it, are projected upon the horizontal plane of projection in lines equal and parallel to themselves ; and the same remark applies to the projections of the line CF, and to all lines parallel to it, upon the vertical plane of projection.

The line CF, and all lines parallel to it, are perpendicular to the horizontal plane of projection, and are projected on that plane in points. Also, the line BC, and all lines parallel to it, are perpendicular to the vertical plane of projection, and are projected on that plane in points.

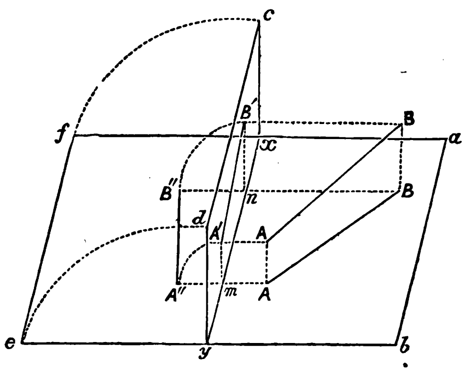
NOTE 1.—When a line is parallel to the horizontal and vertical plane, its projections are lines parallel to xy, the line of intersection, and equal in length to the original line. The projections C', D' and C, D, of the line CD are parallel to xy, and equal in length to CD.

NOTE 2.—When a line is perpendicular to the plane of projection its projection on that plane is a point. The lines CB, CF, respectively perpendicular to the vertical and horizontal plane, are projected on those planes in the points $C'C$.

NOTE 3.—Just as when the projections of a point are given, the point itself may be found, so the preceding solid may be determined from its projection on the two planes. For example, the surface BCFG is the intersection of the projecting plane of BC with the projecting plane of $C'F'$. The remaining surfaces are the intersections of the projecting planes of the lines which are the projections of those surfaces.

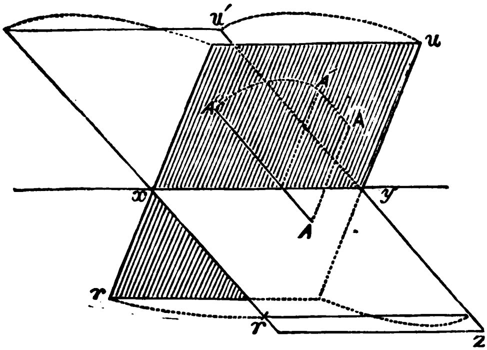
12. Since objects whose surfaces lie in different planes have to be represented upon a sheet of paper which is but one plane, the vertical plane must be supposed to revolve upon the line of intersection of the planes of projection until it coincides with the horizontal plane.

13. Thus in the figure following, the vertical plane *cdxy* after revolving one-fourth of a revolution, as shown by the arcs *ed*, *fc*, will



assume the position *efxy*, which is a continuation of the horizontal plane *abxy*. Now after the vertical plane *cdxy* has revolved as described, the vertical projection A' of the point A will assume the position A''. In the same manner, the vertical projection B' of the point B will assume the position B'', and the line joining A'' and B'' will be the vertical projection of a line in space, whose horizontal projection is AB.

14. Again, in the subjoined figure, let xyu and xyu' represent the two planes of projection, xyu being the vertical, and xyu' the horizontal.



Let the vertical plane xyu revolve about xy until it takes the position xyu', when xyz the horizontal plane and xyu' form one plane. Let A be a point in space; a line drawn at right angles to the vertical plane will intersect it in A', which is the *vertical* projection of the point A. Next, drop a perpendicular from A to meet the horizontal plane xyz in the point A, which will be the *horizontal* projection of the point A. On revolving the planes as stated, the vertical projection A' will assume the position A''.

After the vertical plane has been *turned down* to coincide with the horizontal plane; xyr, *i.e.*, that portion of it which was below the line of intersection, will take the position xyr'. Consequently, any *elevation* on it will be in front of the line of intersection.

15. It will be seen from the foregoing figures, that the distance of the *elevation* of a point from the ground line (xy) shows the distance of the point from the horizontal plane; also that the distance of the *plan* of a point from the ground line (xy) shows the distance of the point from the vertical plane.

NOTE.—In the course of solving problems in *solid geometry* it is frequently necessary to revolve a plane, until it coincides with the plane of projection. This is termed *rabatting*, or constructing the plane.

Section II.

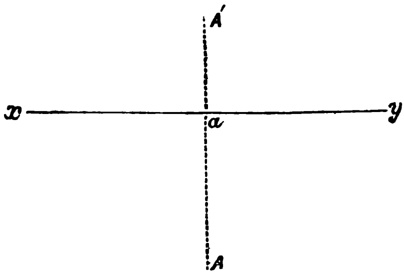
PROJECTION OF POINTS, LINES, &c.

Problem 1.

To find the **plan** *of a* **point,** *its* **elevation** *being given.*

Let A' be the elevation of the given point. It is required to find its plan.

From A' draw a line perpendicular to xy. The plan of A' will be in this line. Let A be its plan. Then the points $A'A$ are the elevation and plan of a point in space, the height of which above the horizontal plane is equal to $A'a$, and the distance of which from the vertical plane is equal to Aa.



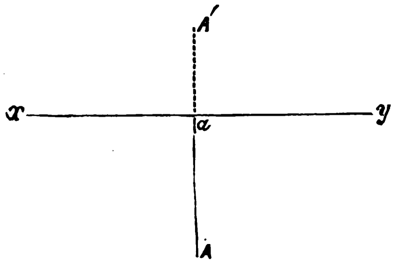
Problem 2.

To draw the **plan** *of a* **line,** *its* **elevation** *being given* **at right angles to the vertical plane.**

As the line is at right angles to the vertical plane, it will be projected on that plane in a point, just as (see figure, page 191) the line BC is projected on *abxy* in the point C'.

Let A' then be its elevation.

The plan of the line is found by dropping a perpendicular from A', and making Aa equal in length to the given line. The line Aa



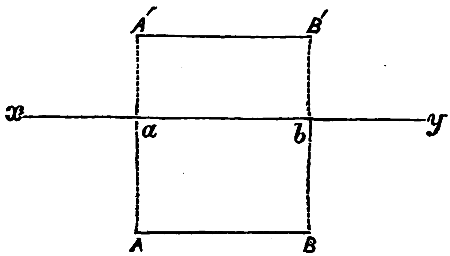
is parallel to the horizontal plane, and is projected on that plane in length equal to the original line.

NOTE.—The line Aa, viewed in the direction of its length, will be seen as the point A' in the vertical plane; and A', viewed from *above*, *i.e.*, at right angles to xy, will be seen as Aa. Thus, Aa and A' are the plan and elevation of a line at right angles to the vertical plane, parallel to, and situated above the horizontal plane at a distance equal to $A'a$.

Problem 3.

To find the **plan** *of a* **line,** *its* **elevation** *being given* **parallel to the two planes of projection.**

As the line is parallel to the two planes, its projections will be parallel to xy. Let $A'B'$ then be its elevation.



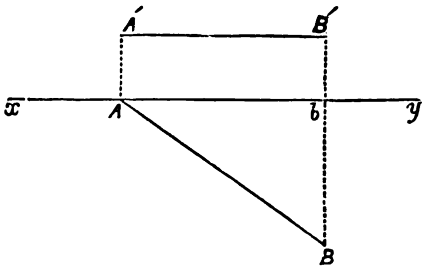
Its plan AB is parallel to xy, and equal in length to the original line. The lines $A'B'$ and AB are the projections of a line in space, elevated above the horizontal plane a distance equal to $A'a$ or $B'b$, and removed from the vertical plane a distance equal to Aa or Bb.

Problem 4.

To find the **plan** *of a* **line,** *its* **elevation** *being given* **parallel to the horizontal plane,** *but* **inclined to the vertical plane of projection.**

Here the elevation of the line is parallel to xy, as in **Pr. 3.**

Let $A'B'$ be its elevation.

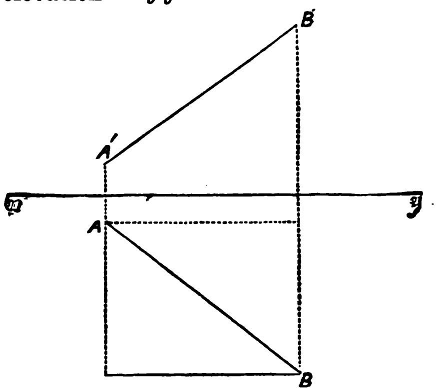


The plan AB shows that the line meets the vertical plane in A, also that it is inclined to that plane at an angle BAb.

The plan AB is the real length of the line.

Problem 5.

To find the **plan** *of a* **line, inclined to both planes of projection,** *its* **elevation** *being given.*

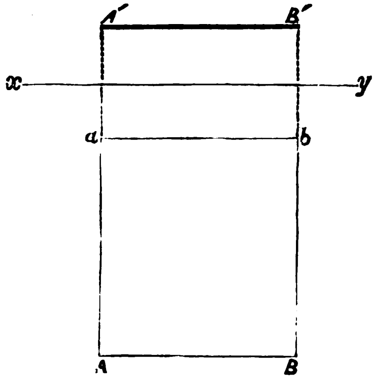


Let $A'B'$ be the elevation. Taking A and B as the distances of points A' and B' respectively, from the vertical plane, the line joining those points will be the plan of $A'B'$.

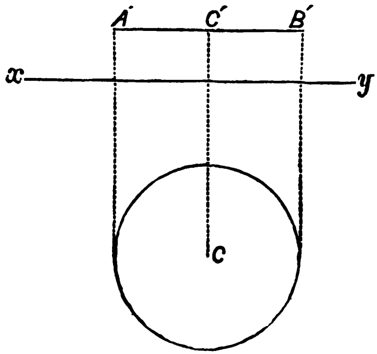
NOTE.—Neither the plan nor the elevation expresses the real length of the line, nor its inclination to the two planes of projection. (See Pr. 4.)

Problem 6.

To find the **plan** *of the* **end elevation** *of a* **rectangular surface** *given* **parallel to the horizontal plane.**



Let $A'B'$ be its elevation. Now, the points A' and B' represent



lines perpendicular to the vertical plane, the projections of which will be found as in **Pr. 2.**

Make *Aa* and *Bb* equal in length to the given surface, and join *AB* and *ab*; then *ABba* is the required plan.

It may be here remarked that the edge view of a circular surface will be a *line* equal in length to the diameter of the circle.

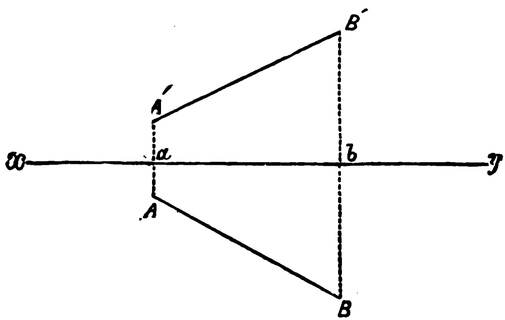
For example, let it be required to find the plan of a circle, having its edge view given as in this problem (see figure, page 197).

Let *A'B'* be its elevation. Then point *C'* will represent the centre of the circle, and the required plan will be found by taking any point *C* in the projector from *C'*. Then from *C* as centre, and radius *C'A'* or *C'B'* describe a circle which will be the required plan.

Problem 7.

To find the **elevation** *of a* **line,** *its* **plan** *being given.*

Let *AB* be the plan of the given line. Draw *AA'* and *BB'* at right angles to *xy*, making *A'a* and *B'b* equal to the supposed height



of *A*, *B* above the horizontal plane, and join *A'B'*; then *A'B'* is the elevation required.

NOTE.—The line is inclined to both planes of projection as in **Pr. 5.**

Section III.

ELEMENTARY SOLIDS.

The solids most commonly used to illustrate the principles of Solid Geometry are as follows: the *cube, prism, pyramid, sphere, cone,* and *cylinder.*

(1.) "A **cube** is a solid figure contained by six equal squares" **(Euc. XI., Def. 25)**.

(2.) "A **prism** is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others parallelograms" **(Euc. XI., Def. 13)**.

(3.) "A **pyramid** is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet" **(Euc. XI., Def. 12)**.

(4.) "A **sphere** is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved" **(Euc. XI., Def. 14)**.

(5.) "A **cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone" **(Euc. XI., Def. 18)**.

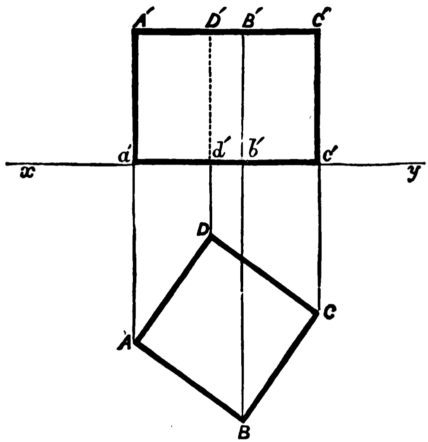
(6.) "A **cylinder** is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed" **(Euc. XI., Def. 21)**.

Problem 8.

To find the **plan** *of a* **cube,** *its* **elevation** *being given.*

Let $A'B'C'D'$, &c., be the elevation of a cube. It is required to find its plan.

From A' let fall a perpendicular to xy, and produce it from a' to A, making $a'A$ equal to the distance that A' is from the vertical plane. From B' draw a line at right angles to xy, and from A, *with a radius equal to a side of the cube* as $A'a'$, cut the perpendicular in B. Join AB; it will be the plan of $A'B'$. Now from C' draw a line perpendicular to xy and produce it. From B, *with a radius equal to*



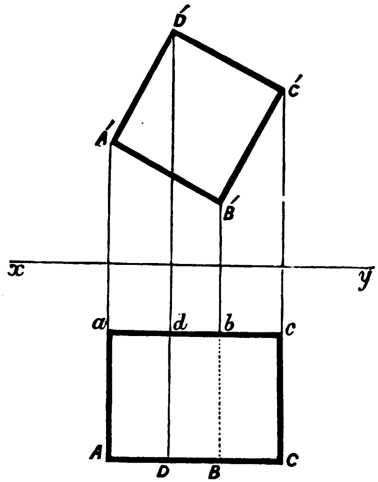
a side of the cube, as $B'b'$, cut it in C, then C will be the plan of C'. Join BC; it will be the plan of $B'C'$. Again, drop a perpendicular from D' to xy, and produce it. From C, *with radius* $C'c'$ describe an arc to cut it in D. Join CD and DA to complete the required plan. The points $a'd'b'c'$ being opposite to $A'D'B'C'$; their plans are exactly covered by the points A, D, B, C.

Problem 9.

The **elevation** *of a* **cube** *being given, when* **one face** *is inclined to the ground at an* **angle** *of* **60°** *and* **another face** *at an* **angle** *of* **30°**, *to find its* **plan.**

Let $A'B'C'D'$ be the elevation of the given cube. It is required to find its plan.

From A' let fall a perpendicular $A'A$, and make Aa equal to the length of the line represented by the point A'; *i.e.*, equal to $A'B'$ or any side of the square $A'B'C'D'$. Through A and a, draw lines paral-



lel to xy, and from D', B' and C', drop perpendiculars intersecting these lines in D,d, B,b, and C,c; for as all the edges of a cube are equal, the lines of which the points $B'C'D'$ are the vertical projections are equal to that expressed by A'; *i.e.*, to Aa.

NOTE.—The plan of the edge of the cube expressed by B' is shown by a dotted line, because it is not seen.

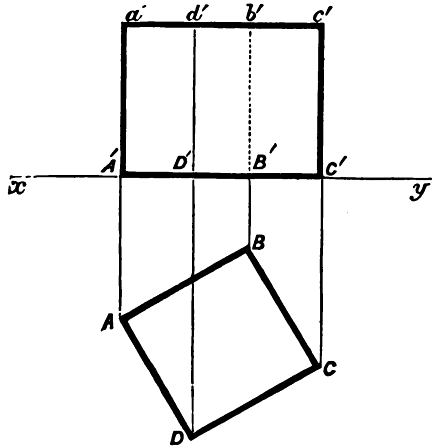
Problem 10.

To find the **elevation** *of a* **cube,** *its* **plan** *being given.*

Let $ABCD$ be the plan of a cube. It is required to find its elevation.

In this case, each *corner* of the square is the plan of *one* of the perpendicular edges of the cube, and $ABCD$ is the plan of the upper surface also. From the points A, D, B, and C, draw projectors at right angles to xy; and at the points where these meet the ground line, we have the elevations of the four corners of the square.

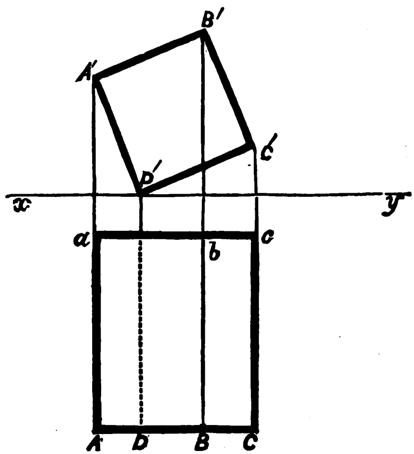
Continue the projectors through *A′*, *D′*, &c., making *A′a′*, *B′b′*, *C′c′*, &c., equal in height to the edge of the given cube.



A line then drawn through the points *a′*, *b′*, *c′*, &c., will be parallel to *xy*, and will complete the required elevation.

Problem 11.

To draw the **plan** *of a* **square prism,** *its* **elevation** *being given.*



Let *A′B′C′D′* be the elevation of a square prism. It is required to draw its plan. From the angular points A', B', C', &c., draw lines at right angles to the line of intersection (xy), and upon these set off the length of the prism; *i.e.*, make Aa, &c., equal to the length of the solid. Through the points A and a draw lines parallel to xy, to show the ends of the prism. Each of the points A', B', C', &c., represents a line perpendicular to the vertical plane. In the plan these lines will be shown at right angles to xy (**Pr. 2**).

NOTE 1.—The line represented by D' will be dotted in the plan, because under the given circumstances it is not seen.

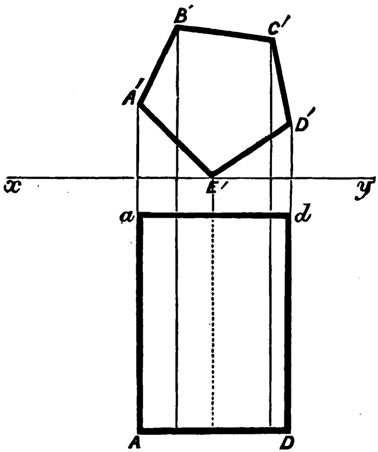
NOTE 2.—The ends of prisms may be triangles, squares, or polygons; a prism is said to be triangular when its ends are triangles, square when its ends are square, &c., &c.

NOTE 3.—The axis of a prism is a line joining the centres of the bases.

NOTE 4.—When the base is a regular figure, it is called a regular prism but when the base is an irregular figure, the solid on it is termed irregular.

Problem 12.

To draw the **plan** *of a* **pentagonal prism,** *its* **elevation** *being given.*



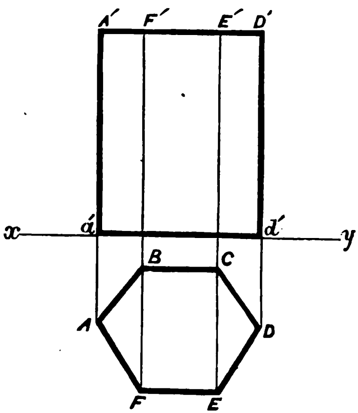
Let $A'B'C'D'E'$ be the elevation of a pentagonal prism. It is required to draw its plan. From the angular points, A', B', C', &c., draw lines at right angles to the ground line (xy), and upon these set off the length of the prism; *i.e.*, make Aa, &c., equal to the length of the solid. Through the points A and a, draw lines parallel to xy to show the ends of the prism. Each of the points A', B', C', &c., represents a line perpendicular to the vertical plane. In the plan these lines will be shown at right angles to xy (**Pr. 2**).

NOTE.—The line represented by E' will be dotted in the plan, because under the given circumstances it is not seen.

Problem 13.

To find the **elevation** *of a* **hexagonal prism,** *its* **plan** *being given.*

Let $ABCDEF$ be the plan of a hexagonal prism. It is required to find its elevation.



From A, raise a perpendicular to xy, and from a', the point in which it cuts xy, set off $a'A'$ equal to the length of the prism. From A', draw $A'D'$ parallel to xy, and from F, E, D, raise perpendiculars to meet this line. We have then drawn the elevation of the hexagonal prism.

Problem 14.

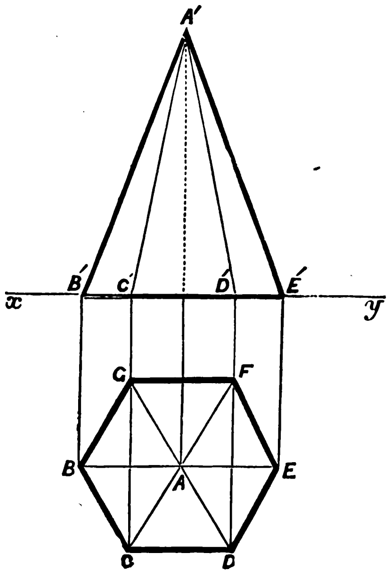
To find the **plan** *of a* **hexagonal pyramid,** *its* **elevation** *being given.*

Let $A'B'C'D'E'$ be the elevation of a hexagonal pyramid. It is required to find its plan.

First, from C' draw a projector $C'C$ perpendicular to xy, also from D' draw a projector $D'D$ at right angles to xy and *equal in length* to $C'C$. Join C and D, and it will be the projection of the *front* edge of the base of the pyramid.

Next, from B' drop a perpendicular to xy, and from C, *with CD as radius*, describe an arc cutting the perpendicular last drawn in B. Join B and C, and it will be the plan of the *side edge* $B'C'$.

Again, from E', drop a perpendicular to xy, and from D, *with DC as radius*, describe an arc cutting the perpendicular last drawn in E. Join D and E, and it will be the plan of the *side edge* $D'E'$.



Now, as the points immediately behind C' and D' are covered *in the elevation* by C' and D', it follows that their projections will coincide; therefore, from B and E as centres, *with radius BC or DE*, cut those projectors in G and F respectively. Join BG, GF, and FE; then these will be the plans of the *remaining edges* of the base. Now, join BE, a projector drawn from A' to meet it will give the point A, the plan of the apex of the pyramid, so that BA and AE are the plans of the lines $B'A'$, $A'E'$. Draw the lines CF and

DG passing through point *A*, so that *CA*, *AD*, will be respectively the plans of *C'A'* and *A'D'*.

Note 1.—*AG* and *AF* are the plans of the lines not seen in elevation.

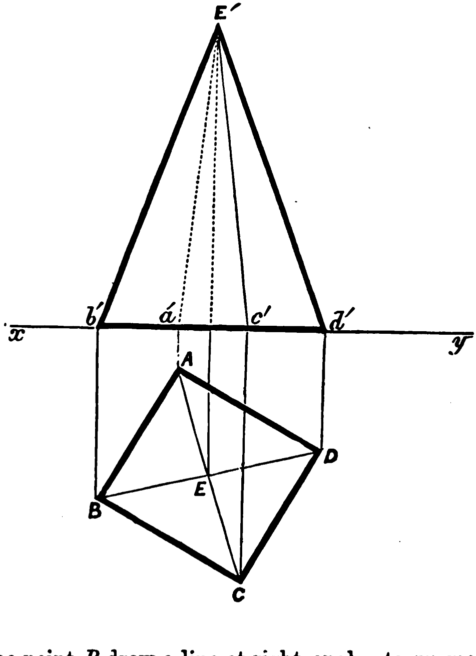
Note 2.—Pyramids take their names from their bases, like prisms.

Note 3.—The axis of a pyramid is a line joining the centre of its base to the apex.

Problem 15.

To find the **elevation** *of a* **square pyramid,** *its* **plan** *being given.*

Let *ABCD* be the plan of a square pyramid. It is required to find its elevation.



From the point *B* draw a line at right angles to *xy*, meeting it in *b'*, also from *C*, *D*, *A*, draw lines perpendicular to *xy*, meeting it in the points *c'*, *d'*, *a'*.

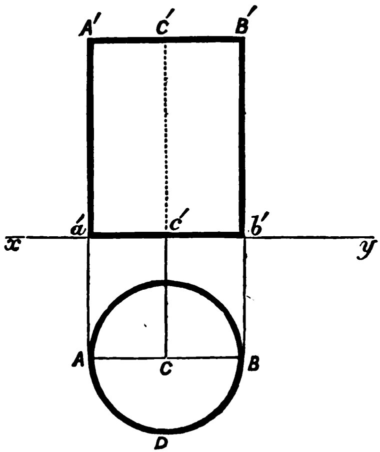
Now, since *C*, *D*, *A* are points on the horizontal plane, and as *xy* represents that plane seen in elevation, the points *b′*, *a′*, *c′*, *d′* on it are the elevations of those points. *E* is the plan of the apex of the pyramid, to find the elevation of which we draw a line at right angles to *xy* from *E*, and elevate it above the line the required height of the point *E′* above the horizontal plane. Join *E′b′*, *E′c′*, *E′d′*, for the angles of the figure.

NOTE—The line *E′a′* being covered by the surface *b′c′E′* is represented by a dotted line.

Problem 16.

To find the **plan** *of a* **cylinder,** *its* **elevation** *being given.*

Let *A′B′a′b′* be the elevation of a cylinder. It is required to find its plan.



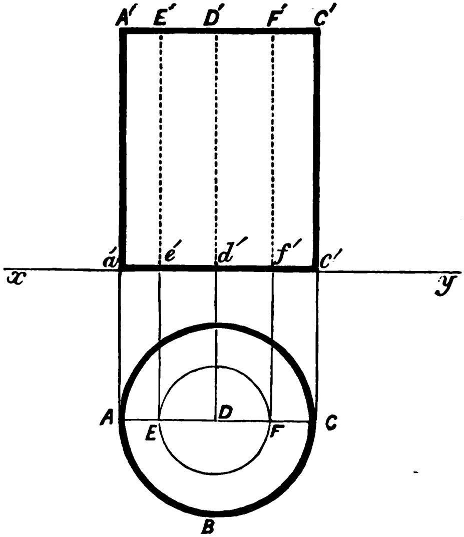
In this case, the surface represented by the line *A′B′* is a circle whose plane is parallel to *xy;* the points *A′* and *B′* will represent the diameter and *C′* the centre. Draw projectors from *A′*, *B′*, and *C′*, to the points *A*, *B*, and *C*. Draw *AC*, *CB*, parallel to *xy*. From *C*, *with radius CA or CB*, describe the required circle *ABD*.

NOTE.—A cylinder may be defined as a prism having an infinite number of faces.

Problem 17.

To find the **elevation** *of a* **hollow cylinder,** *its* **plan** *being given.*

Let ABC be the plan of the given cylinder. It is required to find its elevation.



From A, draw AA' at right angles to xy and making $A'a'$ equal to the length of the cylinder, then draw $A'C'$ parallel to the ground line (xy) and draw Cc' perpendicular to xy.

Let D be the plan of the axis of the cylinder. Now, in order to represent the interior of the given cylinder, draw AC passing through D and parallel to xy. From the points E and F, erect perpendiculars EE' and FF'; then the lines $E'e'$ and $F'f'$ being the elevations of E and F are covered by the surface ABC, and thus are represented as dotted lines in the figure.

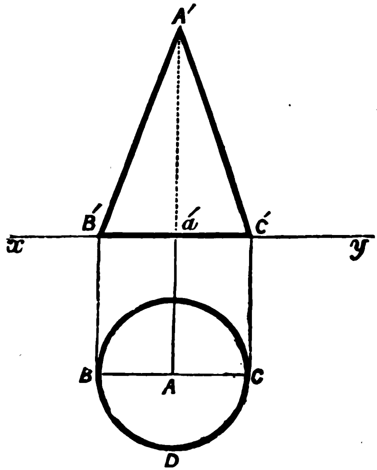
NOTE.—The line $D'd'$ is called the *axis* of the cylinder. When the axis is perpendicular to the plane of its base, the cylinder is termed a *right* cylinder; but when it is inclined to the base, it is termed *oblique.*

Problem 18.

To find the **plan** *of a* **cone,** *its* **elevation** *being given.*

Let $A'B'C'$ be the elevation of a cone. It is required to find its plan.

The line $B'C'$ represents a circle on the horizontal plane, and the points B' and C' will be the extremities of the diameter, and a' the centre. Obtain the projection of this line, by drawing a line parallel



to xy, and draw projectors from B', a', and C' at right angles to xy to meet it in B, A, and C. From A, *with radius AB or AC*, describe the circle BCD. The plan of the apex A' coincides with A, the centre of the circle.

Note.—A cone may be defined as a pyramid, having an infinite number of faces.

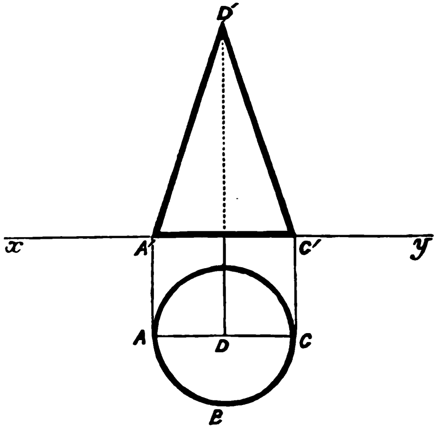
Problem 19.

To find the **elevation** *of a* **cone,** *its* **plan** *being given.*

Let ABC be the plan of a cone. It is required to find its elevation.

From the point A, draw a line at right angles to xy, meeting it in A'; also from point C, draw a line perpendicular to xy, meeting it in the point C'. Now, since A and C are points on the horizontal plane, and as xy represents that plane seen in elevation, the points A' and C' on it are the elevations of those points. D is the plan of the apex of the cone, to find the elevation of which we draw a line at right angles to xy from D, and elevate it above the line the required height of the point D above the horizontal plane.

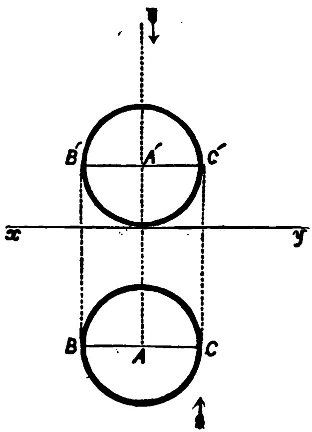
Then join $D'A'$, $D'C'$; and $D'A'C'$ will be the elevation of the cone.



NOTE.—The axis of a cone is a line joining the centre of its base to the apex.

Problem 20.

To find both **plan and elevation** *of a sphere.*



In either case, these will be represented by circles, whose diameters are equal to the diameter of the given sphere ; *e.g.*, take the elevation of the sphere. The view of it from above representing the plan will be a circle of which the line $B'C'$ is the elevation. Again, let the plan be given, the elevation of it will be a circle represented in plan by the straight line BC.

NOTE.—If a semicircle revolve upon its diameter, it generates the surface of a sphere.

Section IV.

TRACES OF LINES AND PLANES.

1. The horizontal and vertical planes of projection are from their mutual relationship termed *co-ordinate* planes.

2. The points in which any line intersects the co-ordinate planes are called the *traces of that line*, and these traces are termed horizontal or vertical, according as they are referred to the horizontal or vertical plane.

3. The lines in which any plane intersects the co-ordinate planes are termed the *traces of that plane*, and are distinguished as the horizontal or vertical trace, according to the plane of projection in which it lies.

4. When the traces of a plane are given, the plane itself is given; and when the projections of a line are given, its traces may be found: or, conversely, the traces being given, its projections may be found.

5. It has been stated that the intersection of the planes of projection is called the ground line, or base line (xy). Now, if a plane be not parallel to the ground line, it must meet it in a point common to both of its traces.

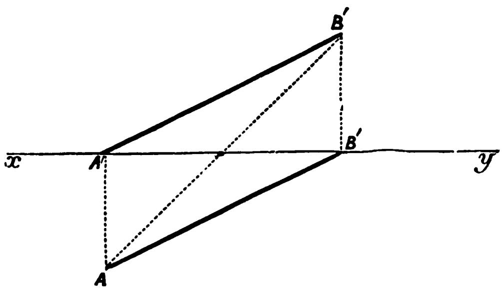
6. If a plane be parallel to the ground line, its traces are also parallel to the ground line; for as the base line is parallel to the plane, it cannot meet it, and therefore cannot meet the traces which are lines in the plane; but each trace and the ground line are in one plane, consequently they are parallel (**Euc. I., Def. 35**).

7. If a plane be perpendicular to the ground line, its traces are also perpendicular to it (**Euc. XI., Def. 3**); and if a plane be parallel to one plane of projection, its trace upon the other is parallel to the ground line (**Euc. XI., 16**).

Problem 21.

Given the **traces** *of a* **line,** *to find its* **projections.**

Let AB' be the traces of the given line; *i.e.*, A and B' are the points where a line in space meets the planes of projection.



The line joining A and B' will be the line in space.

First, let us find its *horizontal* projection. Draw $B'B'$ at right angles to xy and join AB'; then AB' is the projection upon the horizontal plane of the line in space.

Secondly, let us find its vertical projection. Draw AA' at right angles to xy and join $A'B'$; then $A'B'$ is the projection upon the vertical plane of the line in space.

NOTE 1.—Neither plan nor elevation expresses the real length of the line.

NOTE 2.—The vertical trace B' shows that B' is elevated above the horizontal plane a distance equal to $B'B'$, and that A shows that A' is removed from the vertical plane a distance equal to $A'A$.

Problem 22.

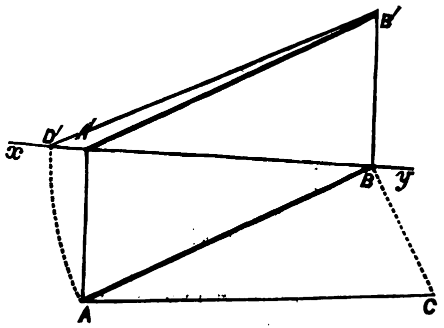
Given the **projections** *of a* **line,** *to find its* **length.**

Let AB and $A'B'$ be the projections of the given line. It is required to find its length.

If we conceive a vertical plane as passing through AB, this plane will have AB for its horizontal trace, and BB' for its vertical. Imagine then this plane to revolve upon AB, until it coincides with the horizontal plane. In order to illustrate what is meant by the plane revolving upon AB, let a triangle be placed with the bevelled edge on AB, and keeping this edge in contact with the surface of the paper, *let the triangle be turned down*, until it becomes horizontal. It will thus assume the position of the triangle ABC. In order to construct this triangle, draw BC perpendicular to AB, and make it equal to BB' (because BB' expresses the height of B above the horizontal plane of projection) and join AC. Then AC the hypotenuse is the real length of the line.

AC may be regarded as the elevation of AB, when viewed at right angles to the plane, passing through it at right angles to the horizontal plane, *i.e.*, in the direction of CB.

The construction may also be made in the vertical plane as follows:—make $B'D'$ equal to AB, and join $B'D'$. Then $B'D'$ is the real



length of the line. The triangle $B'D'B$ represents the vertical plane conceived to pass through AB, after it has been made to coincide with the vertical plane of projection, by being moved through the arc AD'.

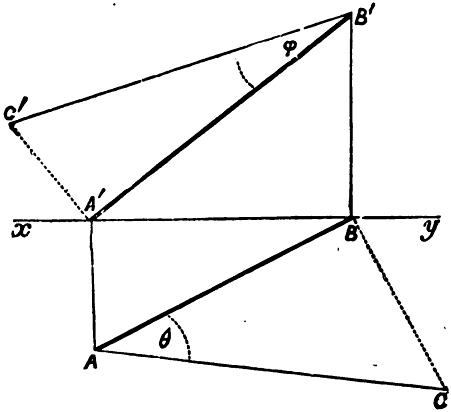
We thus have this practical rule for finding the real length of a line, whose projections are given—viz., upon the given horizontal projection construct a right-angled triangle of which the altitude or perpendicular is equal to the difference of the altitudes of the extremities of the line above the plane of projection. With reference to the vertical plane, we should make the vertical projection of the line the base of a right-angled triangle of which the perpendicular is equal to the difference of the distances of the extremities of the line from the vertical plane of projection.

Problem 23.

Given the **projections of a line,** *to find the* **angles** *which it makes with the* **planes of projection.**

Let AB and $A'B'$ be the projections of the given line. It is required to find the angles which it makes with the planes of projection.

From a consideration of the preceding, it will be readily seen that the angle made with the horizontal plane is *CAB*. The angle made with the vertical plane is *A'B'C'*, which is found as follows :—From point *A'*, *A'C'* is drawn at right angles to *A'B'*, and equal to *A'A*. *B'C'* is then joined.



NOTE 1.—The angle which a line makes with its plan is its inclination to the horizontal plane, and the angle which a line makes with its elevation is its inclination to the vertical plane.

NOTE 2.—When the actual number of degrees is not required, the inclination of a line to the horizontal plane is usually indicated by the Greek letter θ, and to the vertical plane by φ.

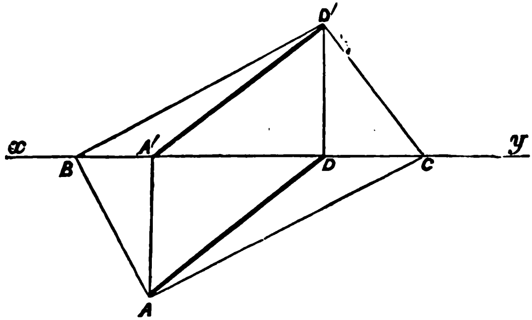
Problem 24.

The **traces of two planes** *being given, to find the* **projections** *of their* **common intersection.**

Let *AB* and *AC* be the horizontal traces of the two planes, meeting in *A ;* and *D'B*, *D'C*, the vertical traces of the two planes, meeting in *D'*. It is required to find the projections of their common intersection.

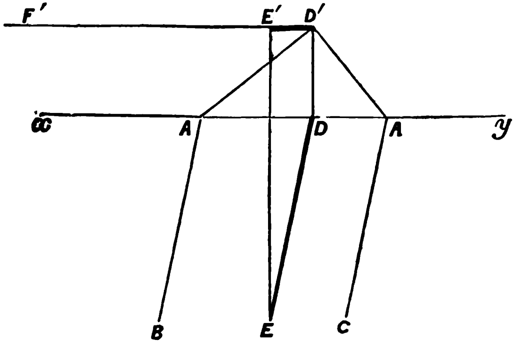
Since the points *A* and *D'* are common to the two planes, the line joining them will be the line in which the planes intersect ; and the projections of this line will fulfil the conditions given.

Now as *A* and *D'* are the traces of a line in space, its projections can be found by **Pr. 21.** Hence, draw *D'D* at right angles to *xy*, and join *AD;* then *AD* is the *horizontal* projection. Similarly, to



find the *vertical* projection. Draw *AA'* at right angles to *xy*, and join *A'D'*; then *A'D'* will be the required vertical projection.

Note.—When the horizontal traces *AB* and *AC* are *parallel*, the hori-



zontal projection of their common intersection, *DE*, will be parallel to *AB*, *AC*; and *F'D'*, its vertical projection, will be parallel to *xy*.

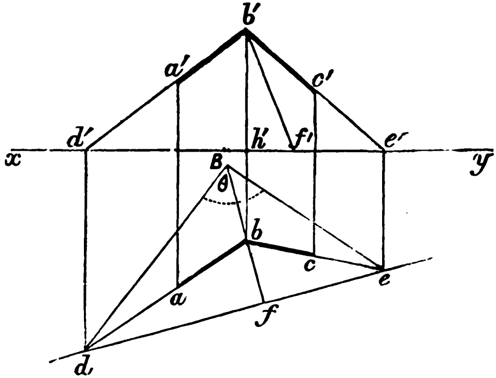
Problem 25.

To determine the **angle** *contained by* **two straight lines,** *AB and BC, given by their* **projections.**

In this case, if the horizontal traces of the lines be joined, a third line will be formed, which, with the two given lines, will form a triangle, the vertical angle of which it is required to determine. If the triangle be constructed into the horizontal plane, its base being the axis of rotation, its true shape will be determined, and therefore the required angle between the lines *AB* and *BC*.

Find *d* and *e*, the horizontal traces of *AB* and *BC*, and join *de*; then *dBe* will be the triangle mentioned. In constructing it into the horizontal plane, *d* and *e* will be fixed, and the point *B* will travel in a vertical plane, at right angles to *de*.

Through *b* draw *bf* at right angles to *de* and produce it beyond *b*. The actual distance of *B* from *f* is the length of the hypotenuse of a



right-angled triangle, of which *bf* is the base, and *hb′* the perpendicular.

Along *xy*, from the point *h*, set off *hf′* equal to *bf*. Join *b′f′*, and make *fB* in the plan equal to *f′b′*. Join *Bd* and *Be*, and the angle *dBe* is that between the two given lines.

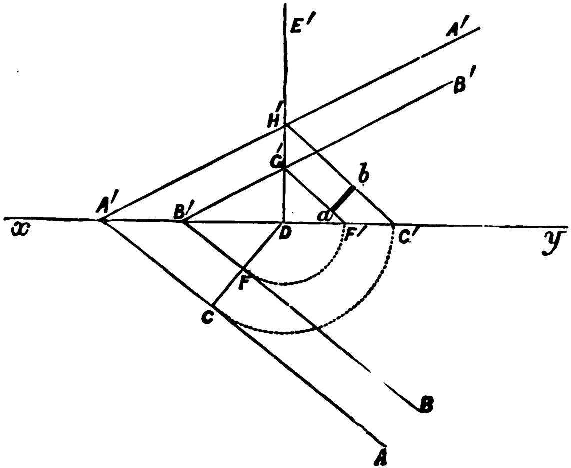
Problem 26.

The **traces of two parallel planes** *being given, to find the* **distance** *between them.*

Let *A′A′*, *AA*, and *B′B′*, *BB*, be the traces of the given parallel planes. It is required to find the distance between them.

Draw *CD* at right angles to the horizontal traces of the planes, and *DE′* perpendicular to *xy*; *CD* and *DE′* will be the traces of a plane at right angles to the horizontal plane of projection.

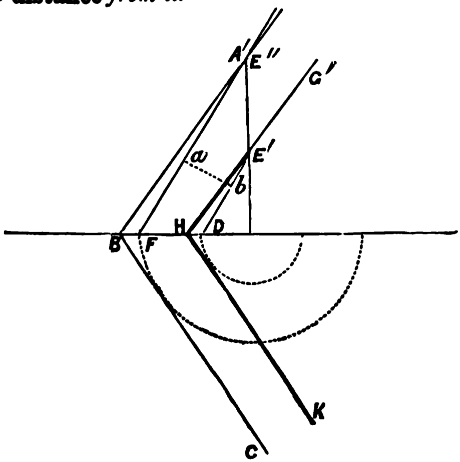
Now this third plane will cut the given planes in two straight lines, which will be parallel to each other; for if two parallel planes be cut by another plane, their common sections with it are parallel. The plane *CD*, *DE′*, cuts *AA′*, *A′A′*, in the line *CD*, and *BB′*, *B′B′*, in the line *DF*. Now we proceed to find the angles which the given planes make with the horizontal plane of projection; we shall then obtain the distance required. Thus, make *DF*, *DC'*, equal respectively to *DF*, *DC*, and join *G'F*, *H'C'*. Then *ab* at



right angles to these lines is the required distance between the planes.

Problem 27.

To determine by its traces a **plane parallel to a given plane,** *and at a given* **distance** *from it.*



Let the given plane be *A'BC*, and *ab* the given distance.

It is required to determine by its traces a plane parallel to *A'BC*.

In this case we proceed as if to find the inclination of *A'BC*, and at a perpendicular distance equal to *ab*, draw *DE'* parallel to *FE''*.

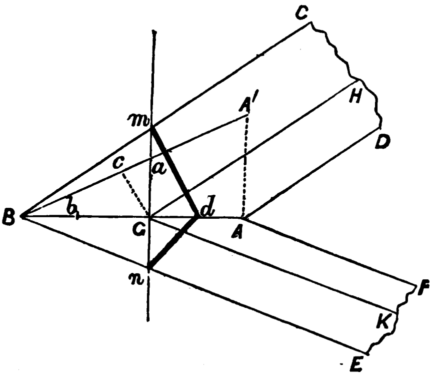
Then *E'* is one point in the vertical trace of the required plane, and as parallel lines have parallel traces, *G'H* and *HK* drawn parallel to *A'B*, *BC*, will be those of the plane required.

Problem 28.

To determine the **angle** *between* **two planes.**

Let *ABCD* and *ABEF* be the two given planes, of which *BC* and *BE* are the traces; and *GH*, *GK*; *AD*, *AF*, horizontals.

First, find the elevation of *AB*, the intersection of the planes; that is to say, make *AA'* drawn perpendicular to *AB* equal to the height of *A* above the plane of projection, and join *A'B*. Draw *mn* at right angles to *AB*, and consider it as the trace of a plane cutting the traces of the given planes in *m* and *n*.



Next, if we consider *mn* as the trace of a plane at right angles to the horizontal plane, it would cut the given planes in a triangular section. This triangle will be found thus—make *Gb* equal to *Ga* (because *G* is elevated above the plane of projection a distance equal to *Ga*) and join *mb*, *nb*.

The solution of the problem then consists in finding the sections of the planes when cut by a third plane at right angles, not to the horizontal plane, but to *AB*, the intersection of the planes. Hence, draw *Gc* at right angles to *A'B*; *Gc* will be the elevation of the plane drawn perpendicular to *A'B*; and as this plane contains the lines which measure the angle between the given planes, we have only to construct *Gc* to find this angle. Make *Gd* equal to *Gc* and join *md*, *nd*; then *mdn* is the angle between the planes. The angle *mdn* is called the *dihedral* angle and *profile* angle of the planes.

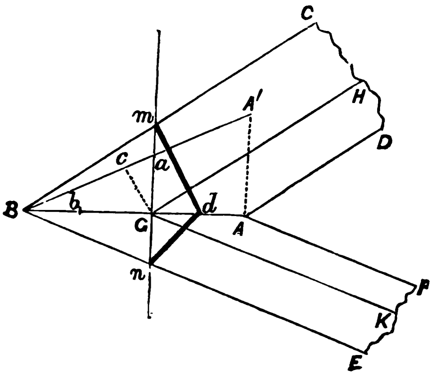
NOTE 1.—A *dihedral angle* is the angle contained by two intersecting planes.

NOTE 2.—The *profile angle* of two planes is the angle contained by the two straight lines in which these planes are cut by a third plane, at right angles to both of them. This third plane is called a *profile plane*. Since these lines are perpendicular to the intersection of the two given planes (**Euc. XI., Def. 3**), the profile angle will be the measure of the dihedral angle (**Euc. XI., Def. 6**).

Problem 29.

To draw a **plane,** *so that it makes a given* **angle** *with a given* **plane** *and* **passes through a line in the first.**

Let *ABEF* be the given plane, the line in which the planes are to intersect each other being *AB*.



Find *A'B*, the elevation of *AB*. Then draw *mn* perpendicular to *AB*, intersecting it in *G*, and from *G* draw *Gc* perpendicular to *A'B*.

Make *Gd* equal to *Gc*, and join *dn*. At the point *d* in *dn*, make the angle *ndm* equal to the angle which the required plane is to make with the given plane *ABEF*.

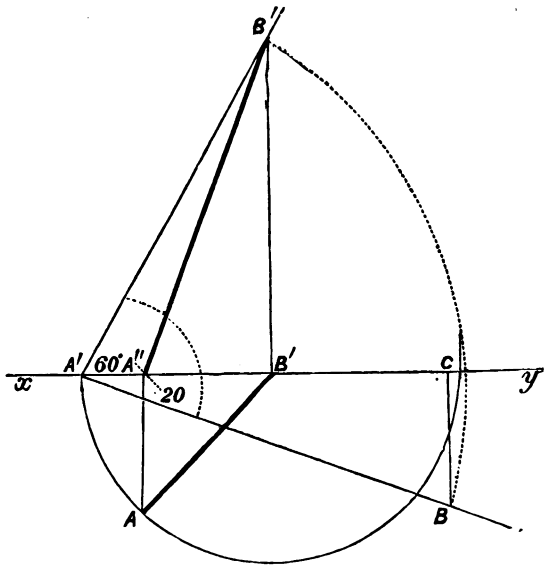
Then the point *m*, where *dm* meets *mn*, will be a point in the horizontal trace of the plane sought; and as *B* is another point in this trace, join *Bm*, and produce it to *C*, then *BC* is the horizontal trace of the required plane, which will be completed by drawing *AD* parallel to *BC*.

Note.—This problem is the converse of the preceding.

Problem 30.

To determine the **plan** *and* **elevation** *of any* **line inclined** *at* **60°** *to the* **horizontal** *plane and* **20°** *to the* **vertical** *plane.*

If a number of lines lie upon the surface of a cone standing with its base upon the paper, and each line having one of its extremities



in the apex, these lines will all be equally inclined to the horizontal. They will also make with the paper the same angle which the sides of the cone makes with its base. The surface of a cone whose base angle is 60° is the locus of all straight lines which pass through the apex, and are inclined at that angle to the horizontal plane.

In *xy* take any point *A'*, draw a line *A'B'* making the angle of 60° with it. Draw *B'B'* at right angles, and consider *A'B'B'* as half elevation of a cone. Then an arc, having *B'* for its centre, and *B'A'* radius, will represent part of its plan.

Now, if lines through *B'* be conceived to lie upon the surface of the cone, only two of them—*i.e.*, those on the extreme right and left—will be shown in their full length in elevation. As the line travels round the solid, its elevation alters its length; when it is in such a position as this, it makes an angle with the vertical plane. In the given case, therefore, we have to determine the exact position upon the cone, when the line is inclined 20° to the vertical plane.

There will be four solutions—two when the line is in front of the cone, and two when it is behind.

At *A'*, set out a line *A'B* equal to the side of the cone, and making an angle of 20° with *xy*. Draw *BC* at right angles to the base line, and the length *A'C* is that of the elevation of the line when it makes an angle of 20° with the vertical plane. With *B* as centre, and radius equal to *A'C*, describe an arc intersecting *xy* in *A''*. Join *A''B'*, which is the required elevation.

A projector from *A''* meeting the arc first drawn in *A*, gives the plan of *A''*, one of the extremities of the line. Join *AB'* and the required problem is solved, *i.e.*, *AB'* is the plan of the line.

Problem 31.

Given an **equilateral triangle** *with* **two** *of its* **sides inclined,** *at* **60°** *and* **30°** *to the horizon, to draw its* **plan**, *and determine the* **inclination of the plane** *in which it is situated.*

Let *ABC* be the given equilateral triangle.

First, draw *BD*, *CE* making with *AB* and *AC* angles of 60° and 30°.

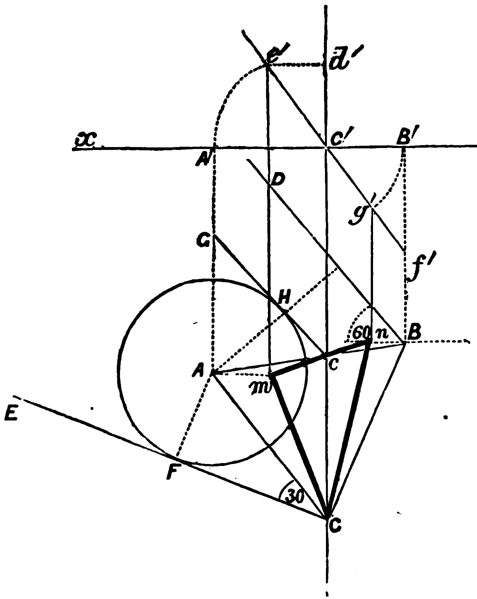
From *A*, draw *AF* perpendicular to *EC*, and with centre *A*, *and radius AF*, describe a circle. Draw *Gc* parallel to *DB*, tangential to this circle, and cutting *AB* in *c*. Then as *AH* is equal to *AF*, *C* and *c* will have the same altitude above the horizontal plane. Let *Cc* be joined, and we have one of the horizontals of the plane sought.

Draw *xy* perpendicular to *Cc* produced, and cutting it in *C'*.

Project the point *A* to *A'*, and with centre *C'*, *and radius C'A'*, describe an arc *A'e'*.

Next, as A is elevated above Cc a distance equal to AH or AF, the point A' must revolve until it is this distance above xy, for the plane revolves upon the horizontal Cc, which is represented in elevation by C'. Therefore make $C'd'$ equal to AH or AF, and draw $e'd'$ parallel to xy, cutting the arc in e'. Join $e'C'$ and produce it to f', then $e'f'$ is the elevation of the plane containing the triangle.

Further, project B to B', and with centre C', *and radius* $C'B'$, describe an arc, cutting $e'f'$ in g'. Through A and B draw unlimited lines parallel to xy, and from points e' and g' draw $e'm$, $g'n$, cutting



these lines in m and n. Join mn, nC, Cm, then mnC is the plan of the triangle when the sides AB and AC are inclined at 60° and 30° respectively.

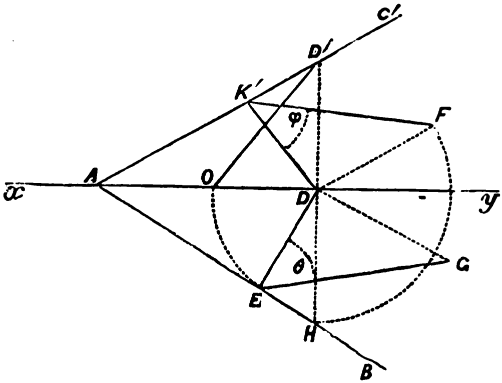
Problem 32.

The **traces of a plane** *being given, to find the* **angles** *which it makes with the* **planes of projection.**

Let AB and AC be the traces of the given plane. It is required to find the angles which it makes with the planes of projection.

First, draw any line *ED* at right angles to the horizontal trace *AB*, and draw *DD′* perpendicular to *xy*, meeting the vertical trace *AC′* in *D′*. Now, since *AC′* intersects the vertical plane, point *D′* will be the vertical trace of a line in space whose horizontal projection is *ED*.

Secondly, the angle which *ED* makes with the horizontal plane may easily be found. This angle is *DOD′* or *GED*. Now, *ED* is the projection of *EG*, and since the line and its projection are drawn



from the same point *E*, which is common to the horizontal plane and the given plane, it follows that *GED* is the angle which the plane makes with the *horizontal* plane of projection.

Thirdly, the angle which the given plane makes with the *vertical* plane of projection is *FK′D*, and it is found by drawing *K′D* perpendicular to *AC′*, and constructing the right-angled triangle *FK′D*, as has been previously done, viz., by drawing *DF* at right angles to *DK′*, and making it equal to *DH*.

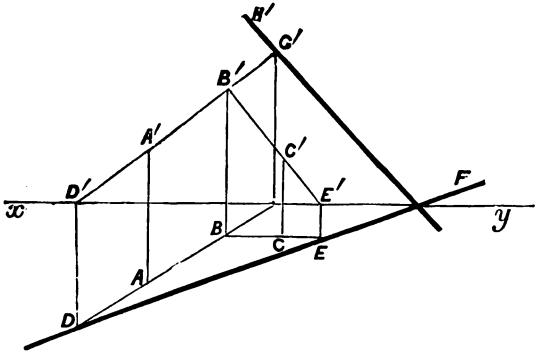
NOTE.—When the traces of a line are situated in the traces of a plane, the line is said to *lie* in the plane. Thus, the line joining *E* and *D* lies in the plane whose traces are *AB* and *AC′*; and when one projection, as *ED*, is given, the other projection may be found.

Problem 33.

To determine by its **traces** *the* **plane** *containing* **three given points**.

Let *A′B′C′* and *ABC* be the projections of the given points. It is required to determine a plane which shall contain them.

Now, if two points are contained by a plane, it is evident that the line joining those two points must also be contained by that plane. Further, if a line be contained by a plane, the traces of that line are in the traces of the plane. The knowledge of these two principles is sufficient for the solution of this problem, for the required plane must contain each of the three lines *AB*, *BC* and *AC*, the traces of which will be points in the traces of the plane.



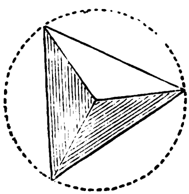
Join *A′B′*, *AB*, *B′C*, *BC*, and produce *B′A′* beyond *A′* to meet *xy* in *D′*. Then a perpendicular to *xy* through *D′*, intersecting the plan of *AB* produced in *D*, gives one point in the required horizontal trace, and *E*, which is the horizontal trace of the line *BC*, is a second point in that trace. The line *DF* drawn through these points is the horizontal trace of the required plane. Find *G′*, the vertical trace of the line *AB*, and draw *H′F* through *G′* to meet *DF* in *F*. Then *HFD* is the plane containing the three given points.

Section V.

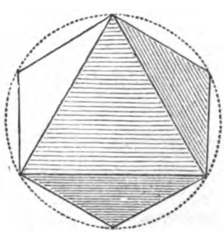
FURTHER PROJECTIONS OF SOLIDS.

In addition to the various solid figures already referred to in the previous sections, there are four other regular solids which must be mentioned, viz.—

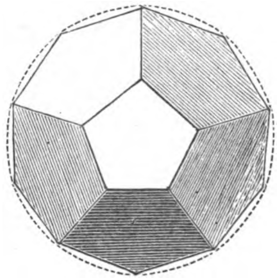
(A) The **Tetrahedron,** which is "a solid figure contained by four equal and equilateral triangles" (**Euc. XI., Def. 26**).



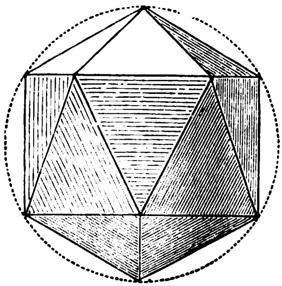
(B) The **Octahedron,** which is "a solid figure contained by eight equal and equilateral triangles" (**Euc. XI., Def. 27**).



(C) The **Dodecahedron,** which is "a solid figure contained by twelve equal pentagons, which are equilateral and equiangular" (**Euc. XI., Def. 28**).



(**D**) The **Icosahedron**, which is "a solid figure contained by twenty equal and equilateral triangles" (**Euc. XI., Def. 29**).



Problem 34.

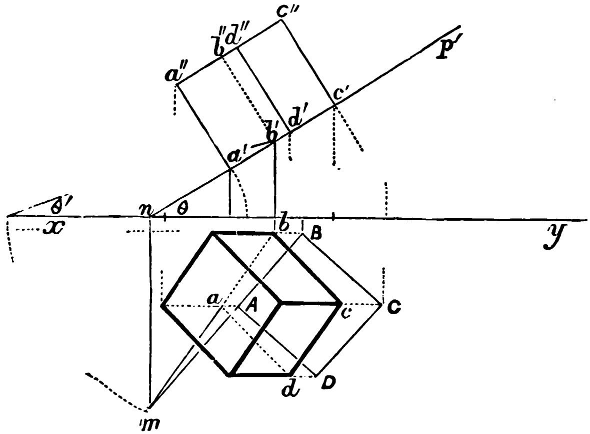
To construct the **projections** *of a* **cube** *having a* **face** *and* **one** *of its* **edges inclined at given angles.**

Let the face *ABCD* be inclined at an angle θ to the horizon, and the edge *BC* at an angle θ'; θ being greater than θ'.

On *xy*, a line of level perpendicular to the trace *mn* of the plane of the base, make an elevation *np'* of this plane; in it place the line *Bm*, whose plan is *bm* inclined at an angle θ'; construct the elevation

$a'b'd'c'$ of the base, by turning the square $ABCD$ through the angle θ, thence find its plan $abcd$.

The elevations of the edges perpendicular to the plane mnp' will be perpendicular to np', and equal in length to the edge of the cube; draw $a'a''$, $b'b''$, $d'd''$, and $c'c''$ at right angles to np', and equal to BC; then join $a''c''$, the figure $a''c'$ will be the elevation; the plans of the



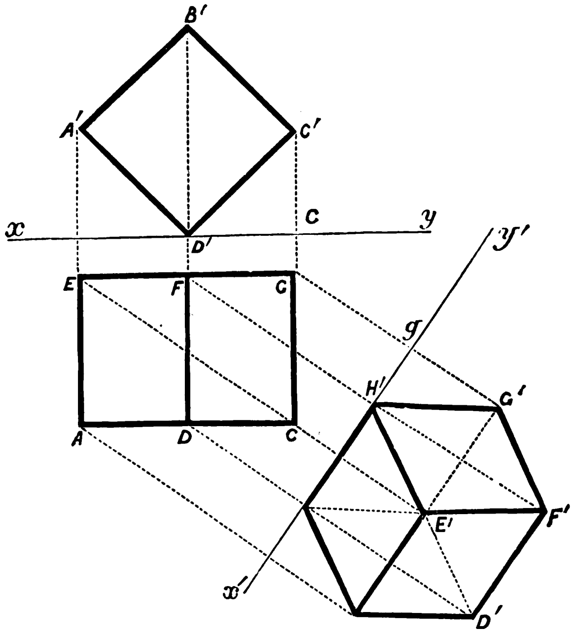
points $a''b'd''c''$ will be the points in which perpendiculars to xy, drawn from these points, cut the parallels to xy drawn from A, B, D, and C.

Problem 35.

To find the **projection** *of a* **cube** *when* **one of its diagonals** *is* **perpendicular** *to the* **plane of projection.**

Construct a square $A'B'C'D'$, and let it represent a face of a cube. Draw xy perpendicular to $B'D'$, one of the diagonals of this face. Then taking $A'B'C'D'$ as an elevation of the cube, its plan will be $ADCEFG$, as explained in **Pr. 9**. Join EC, one of the diagonals of the cube, and draw $x'y'$ at right angles to it. Then the projection of the solid will be obtained as before. For example, the projection of G will be in the projector drawn from G at right angles to xy, and its height above xy is equal to cC'. Therefore make gG' equal to cC'. Also, the projection of F is F' found by setting off

H'F' equal to *D'B'* along the projector drawn from *F*. Join *G'F*. The points *E'*, *D'*, &c., are found in a similar manner.



Problem 36.

To draw the **plan** *of a* **square** *when its* **surface** *is* **inclined 42°,** *and* **one of its sides is horizontal.**

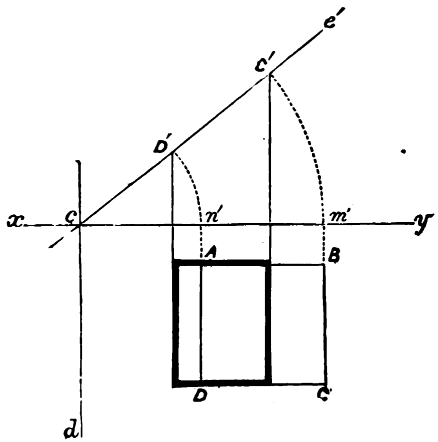
Here as the surface of the square is to be inclined 42°, we commence by assuming the traces of a plane inclined at that angle, and rotate the figure from a horizontal position into this plane.

Then *cd*, the horizontal trace, is perpendicular to *xy*, and the vertical trace makes an angle of 42° with it. The square *ABCD* must be drawn with *AD*, one of its sides parallel to *cd*. Through *C* and *D* projectors must be determined meeting *xy* in *m'* and *n'*.

Then with *c* as centre, describe the arcs *m'C'* and *n'D'*, intersecting the vertical trace in *C'* and *D'*.

These arcs will thus represent the journey of the points *C* and *D* whilst being rotated into the plane *e'cd*. *C'D'* is thus the elevation of the whole square, because *AD* and *BC*, the sides of the square, being horizontal and perpendicular to the vertical plane, their elevations are points.

The intersections of projectors through *D′* and *C′*, with lines



parallel to *xy* through *A*, *B*, *C*, and *D*, are the plans of the corners of the square.

Problem 37.

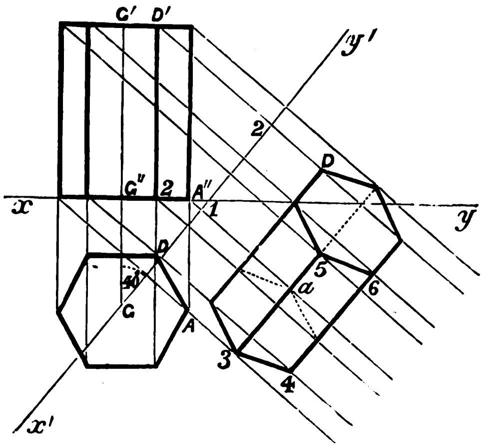
To draw the **plan and elevation** *of a* **hexagonal prism,** *which has its* **axis inclined 40°** *to the* **paper** *and* **one face parallel** *to the* **vertical plane.**

Draw the hexagon with one side parallel to *xy*, which is the plan of the solid when standing with its base upon the paper. By arranging the figure in this way, it will be seen that one face of the object is parallel to the vertical plane. The elevation must be determined from this plan, according to the methods described in the foregoing problems.

Upon the elevation draw *G′G″* to represent the axis of the solid, and produce it. Let us now assume a new *x′y′*, making an angle of 40° with the line *G′G″* produced, the elevation will then be that of the solid, with its axis inclined as regards the new horizontal plane. The student may easily see this by folding his paper upon the new *x′y′*, so as to show both a horizontal plane and a vertical. The first elevation will then be seen under quite a different aspect, viz., that of a solid tilted over.

Now in order to determine the plan, projectors must be drawn through every point of the elevation, at right angles with the assumed $x'y'$, and lengths must be measured along each of these projectors, equal to the distances of the points in the first plan, from the first base line xy. For example, take the point A, a projector $a1$ passes through a at right angles to the new $x'y'$. The distance AA'' is transferred along this projector to the point a beyond the ground line; that is, AA'' is equal to $1a$. In the same manner, all the other points of the base are projected, and thus a new plan of it is obtained.

The plan of the other end of the solid is determined in like manner by projectors through its points in elevation. And as the



first plan is that of both ends, the distances to be measured along the projectors first drawn will be precisely the same as before. To illustrate this, take point D. The distance measured upon the projector beyond $x'y'$ is equal to $2D$.

The whole plan is completed by joining the similar points in each base, as shown in the figure. We may notice, that lines which are parallel in the solid are still parallel, however they may be projected; *e.g.*, 3 4 is parallel to 5 6.

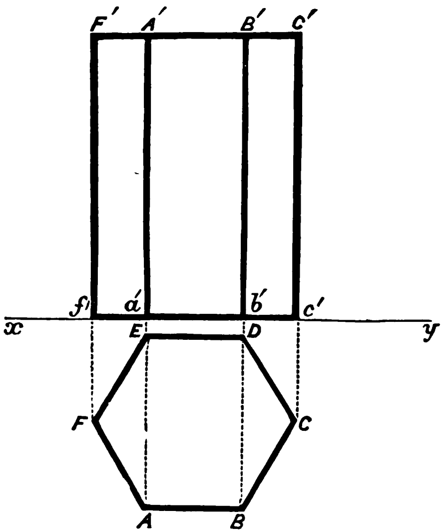
A little consideration of the position of the solid will show that that part of the base in which A is situated is hidden, and that the opposite end is wholly seen in the plan. The edges being dotted indicate this.

Problem 38.

To draw the **plan and elevation** *of a solid* **hexagonal column**, *the height of which is* 30′, *and length of one side* 10′ *on a scale of* 20′ *to the inch; the* **front face** *being* **parallel** *to the* **vertical plane.**

Here the plan will be a regular hexagon. Draw a line *AB* parallel to *xy*, in length half an inch; on it construct a regular hexagon, *ABC*, &c. This will be the required plan.

In order to find the elevation, we draw the projectors *Aa′*, *Bb′*, &c. From *a′b′c′f′* raise perpendiculars *a′A′*, *b′B′*, *c′C′*, and *f′F′*, each $1\frac{1}{2}$



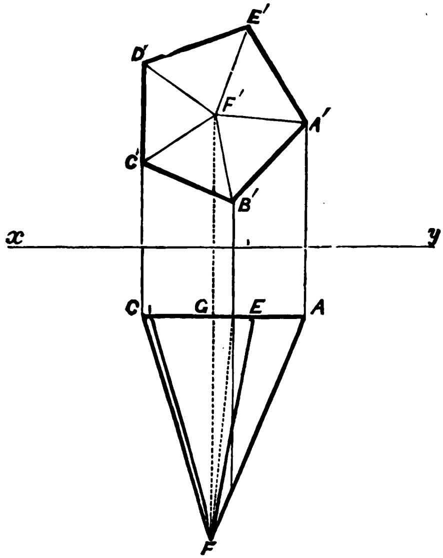
inch in height. Then through *F′*, *A′*, *B′*, *C′*, draw a line parallel to *xy*, and the elevation, consisting of a series of rectangles, will be complete.

It may be remarked that the projectors from *A* and *B*, which are the angles at the base of the front face, are coincident with those that are drawn from *E* and *D*, and are therefore invisible. The rectangle *A′a′b′B′* is an elevation of the front face which rises up from *AB*, and also of the face which rises up from *ED*. Also, the rectangle *F′f′a′A′* is the elevation of the faces which rise up both from *AF* and *FE*; as also the rectangle *B′b′c′C′* is the elevation of faces which rise up both from *BC* and *DC*.

Problem 39.

To draw the **plan** *of a* **pentagonal pyramid** *when one* **edge** *of its* **base** *is* **inclined at an angle of 45°.**

Take a straight line *A′B′*, and let it be inclined to *xy* at an angle of 45°. Then upon *A′B′* construct a regular pentagon *A′B′C′D′E′*,



and join each of the points *A′*, *B′*, *C′*, &c., to *F′* the centre, which is the vertex of the pyramid.

To find the plan of the pyramid; from *F′*, the vertex of the pyramid, draw *F′F*, and set off from *F*, *FG* equal to the height of the pyramid.

The line *FG* represents the axis of the pyramid.

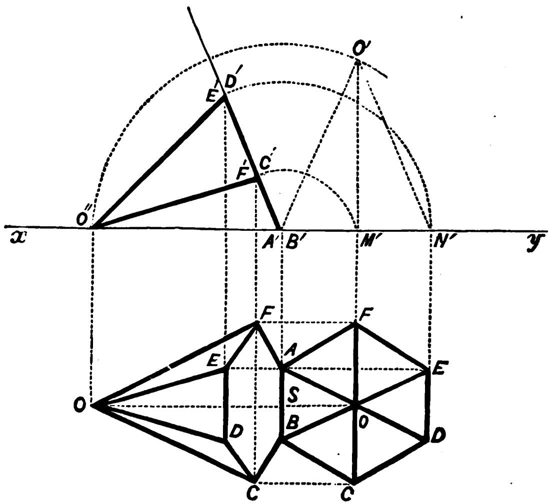
Through point *G*, draw a line parallel to *xy*, and from the points *A′*, *B′*, *C′*, &c., drop projectors, cutting this line in the points *A*, *E*, *C*, &c. Join each of these points to *F*, and we have the required plan of the pyramid, having an edge of its base inclined at 45°.

Problem 40.

To determine the **plan** *of a* **hexagonal pyramid** *when* **lying on one of its faces on the horizontal plane.**

Describe the hexagon *ABCDEF*, find its centre *O*, and join it to each of the angular points *A*, *B*, *C*, &c. This will complete the plan of the pyramid, when its base is horizontal, *O* being its vertex.

Now, the projection of the solid required is that which will result after the pyramid has revolved upon one of the edges of its base, as *AB*, until the face *OAB* rests in the horizontal plane. To obtain



this projection, then, we must first determine the angle which the faces of the pyramid make with its base. The face *OAB* represents a plane whose horizontal trace is *AB*, and we must construct a right-angled triangle to find its inclination, having *OS*, drawn at right angles to *AB*, for its base, and the height of the pyramid, that is, the height of *O* above *A* and *B*, for its perpendicular.

Produce *AB* and draw *xy* at right angles to it. The vertical projection of *BA* is *B′A′* in *xy*. Draw the projector *OO′*, making *O′M′* equal to the height of the pyramid, and join *O′* to *B′A′*. Then, the angle $O'B'M'$ expresses the inclination of the face of the pyramid to its base, and consequently to the horizontal plane.

Make $A'O''$ equal to $A'O'$, and from A' draw $A'E'$, making the angle $O''A'E'$ equal to $O'B'M'$.

Next produce CF, DE, meeting xy in M' and N', and from centre A', with radii $A'M'$, $A'N'$, describe arcs cutting $A'E'$ in F', C', and E', D'.

Join $O''E'$, $O''F'$, which completes the elevation of the pyramid when resting on one of its faces. This will be understood by joining $O'N'$, when $O'A'N'$ would be the elevation of the pyramid resting with its base upon the horizontal plane, and $O''A'E'$ is $O'A'N'$ after the latter has revolved until $A'O'$ assumes the position of $A'O''$.

The plan is thus determined—the plan of C' is C, being the point of intersection let fall from C'' with a line drawn from C parallel to xy. The plan of the other points may be similarly obtained.

Problem 41.

To project a **hexagonal pyramid** *whose* **axis** *is* **inclined at an angle of 50°** *to the* **horizontal plane,** *but* **parallel** *to the* **vertical.**

The projections of the given pyramid having its axis parallel to the vertical plane, and at right angles to the horizontal plane, are shown in the following figure No. 1, its plan being a hexagon, and its elevation an isosceles triangle.

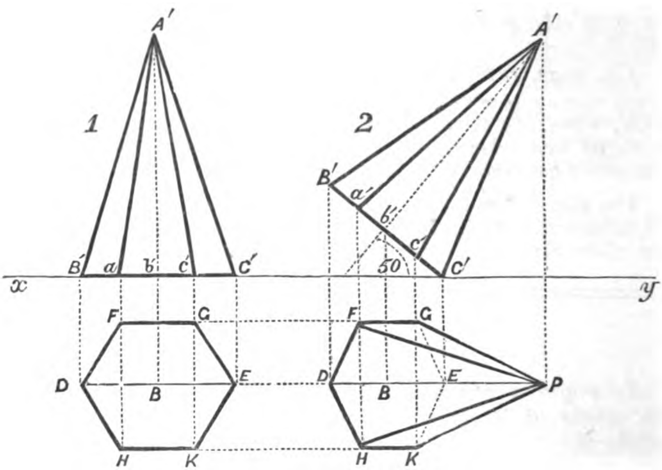
Place elevation No. 1 at the given inclination, for elevation No. 2.

The projection of the various points for the plan of the pyramid are found by letting fall perpendiculars from the various points in the elevation of No. 2 to intersect the parallels to xy, drawn from the corresponding points in the plan of No. 1. Hence the vertical projectors from B', b', C', A', falling on the parallel DE produced, determine the position of these points on the plan. The projector from a', which meets FG and HK produced (No. 1), determines the position of the angles F and H on the plan No. 2. And so the projector from c', which meets FG and HK produced (No. 1), determines the positions of angles G and K in plan No. 2.

Join P, the plan of the apex of the pyramid, with D, H, K, E, G, F, the contiguous angular points in the base, and the plan of the whole pyramid will be complete. The edges of the base GE, EK, are dotted, because they are hidden by the body of the pyramid.

It will be seen that the axis *BP* and the diameter *DE* are in the same line, which is parallel to the vertical plane.

NOTE.—*Four* faces are seen in the plan: *e. g.*, *PFG* is the plan of the face of which *FG* is the plan in No. 1; *PFD* is the plan of the face of



which *FD* is the plan in No. 1; *PDH* is the plan of the face of which *DH* is the plan in No. 1; and *PHK* is the plan of the face of which *HK* is the plan in No. 1.

It will be observed that *two* faces are hidden in the inclined plan, viz., those of which *PGE* and *PKE* are the plans.

Problem 42.

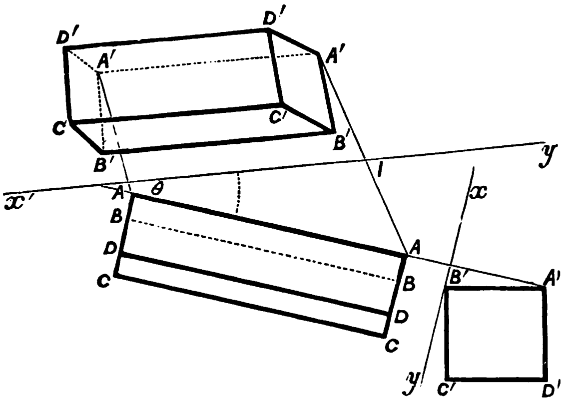
The **projections of any solid** *being given, to determine* **other projections** *from them.*

Let the figure *AC* be the plan of a square prism, of which *A'B'C'D'* is the end elevation. It is required to determine a new elevation upon a vertical plane, making an angle, θ, with the long edges of the solid.

Assume *x'y'* making the required angle with either of the plans of the sides, as *AA*.

Projectors through *A*, *B*, *C*, and *D* in the plan, at right angles to *x'y'*, will contain the required elevations of the points *A'B'C'D'*.

Now, as the heights of these points above the horizontal plane are shown in the given end elevation, it is only necessary to transfer



them from one elevation to the other. For example, the distance AA' is the same as $A'1$.

Similarly, we obtain the elevation of the end A', B', C', D'; and since the edges of the solid are parallel to the horizontal plane, the heights of the corners are the same as those of A, B, C, D.

Problem 43.

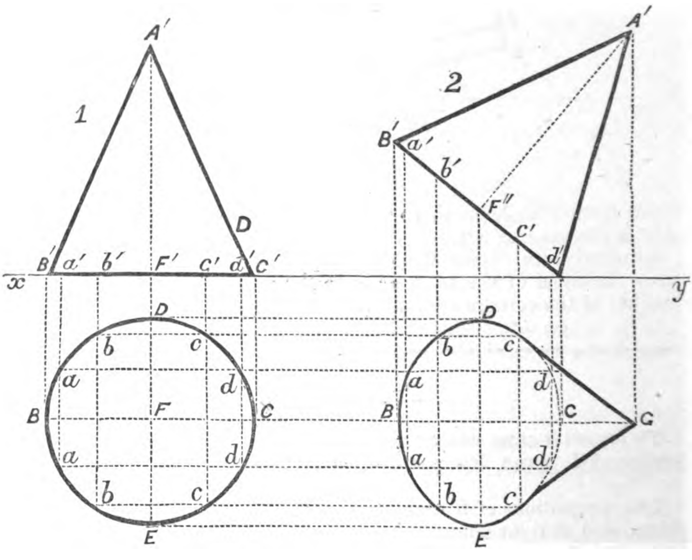
***To project a* cone *whose* axis *is* inclined at an angle of 45° *to the* horizontal plane, *but* parallel *to the* vertical.**

The projections of a cone having its axis parallel to the vertical plane, and at right angles to the horizontal plane, are shown in the following figure No. 1, its *plan* being a *circle*, and its *elevation* an *isosceles triangle.*

Now, in order to project a cone whose axis is inclined at an angle of 45° to the horizontal plane, we draw $B'C'$, the elevation of the diameter of the cone at an angle of 45°; because when the axis is inclined at 45°, the diameter of the base is also inclined at the same angle. Transfer the elevation $B'A'C'$ of No. 1 to $B'A'C'$ of No. 2. Draw a diameter BC on No. 1 parallel to xy, and another DE at right angles to it. Then the length of the diameter BC on plan No. 2 is determined by projectors dropped from B' and C' to meet the diameter BC, No. 1 produced.

The width of the diameter *DE* on plan No. 2 is the same as the width of diameter *DE* on plan No. 1, and is found by parallels to *xy* drawn from *D* and *E*, No. 1, to meet a projector dropped from *F''*, No. 2. It will now be seen that the plan of the circle No. 1 at an angle of 45°, as shown in No. 2, is an *ellipse*, of which *BC* and *DE* are the conjugate and transverse diameters. The points through which the curve of the ellipse passes are thus projected. On plan No. 1, mark off *any* points, *e.g.*, *aa*, *bb*, *cc*, *dd*, on each side of *BC*, and at equal distances from it. Join *aa*, *bb*, *cc*, *dd*, and produce the lines as projectors to *B′C′* in the points *a′*, *b′*, *c′*, *d′*.

Now let these points be transferred to *B′C′*, the elevation of the diameter in No. 2, and from these transferred points drop projectors



to intersect the parallels to *xy* drawn from the corresponding points in plan No. 1. Then the curve of the ellipse passes through the points of intersection, which curve must be traced by hand.

The plan of the apex *A′* is found by projecting *A′G* on to the parallel *BC* produced. Tangents drawn from *G* to the curve of the ellipse, representing the sloping edges of the cone, will complete the plan of the whole cone.

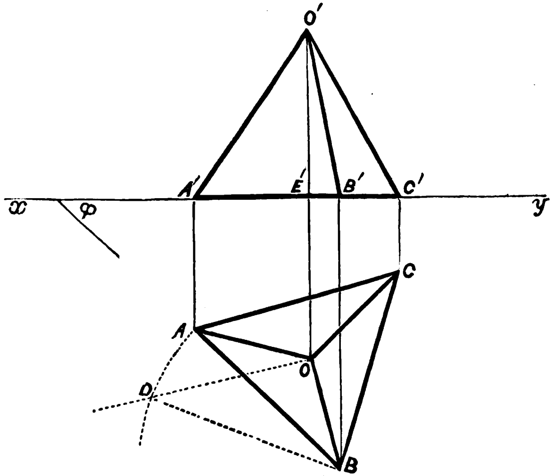
It will be seen that the tangents do not meet the curve in *D* and *E*.

Problem 44.

To construct the **projections** *of the* **tetrahedron.**

In this case we assume the given solid to rest on one of its faces on the horizontal plane.

Let *AB* be one of the edges resting on the horizontal plane, and inclined to the ground line at any given angle φ. On *AB* describe the equilateral triangle *ABC*. Find *O*, the centre of the circum-



scribing circle, and join *AO, BO, CO;* then *O* will be the plan of the vertex, *ABC* will be the plan of the base, and *AO, BO, CO,* will be the plans of the edges meeting in *O.*

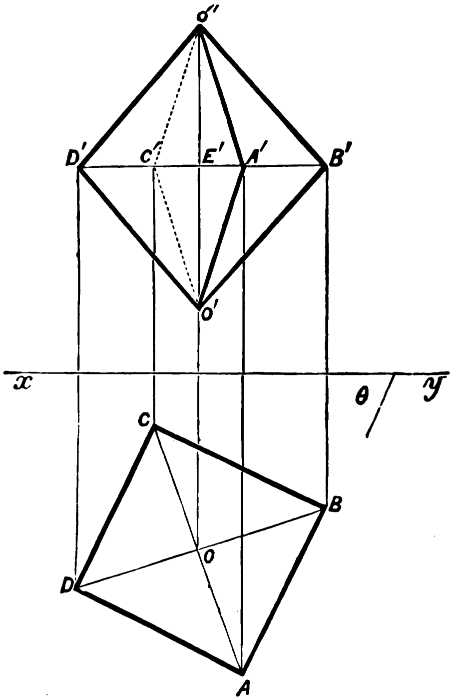
From *O*, draw *OD* at right angles to *OB;* then with *B* as centre, *and BA as radius,* describe an arc cutting *OD* in *D; OD* will be the height of the pyramid. Draw *AA′*, *BB′*, *CC′*, and *OO′* perpendicular to *xy;* make *E′O′* equal to *OD;* then join *A′O′*, *B′O′*, and *C′O′*, and the figure thus formed will be the elevation of the solid.

Problem 45.

To construct the **projections** *of the* **octahedron** *when its* **axis** *is* **vertical.**

In this case, the axis is assumed to be vertical; let *AB* be one edge, and let it make any angle θ with xy. On AB describe the square $ABCD$, and draw the diagonals AC, BD, intersecting in O; then $ABCD$ will be the plan of the octahedron, O being that of the vertex.

Next, from O draw OO'' at right angles to xy, and make $O'O''$ equal to the diagonal of the square $ABCD$; bisect $O'O''$ in E', draw $D'B'$ parallel to xy, and DD', CC', AA', and BB' perpendicular to xy, then



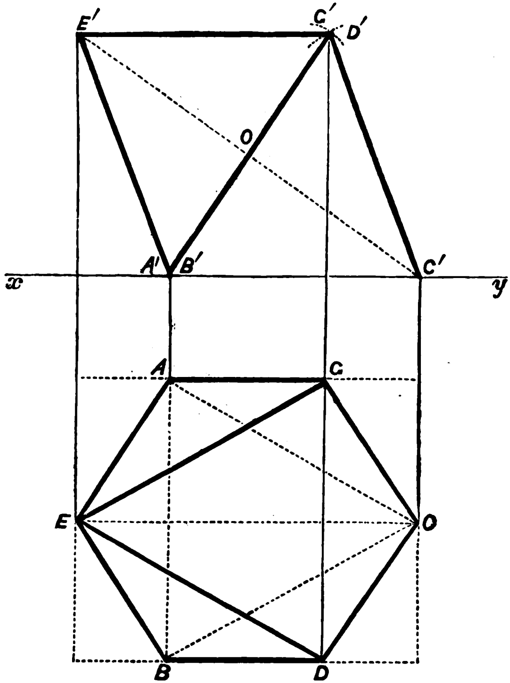
O', O'', A', B', C', D', will be the elevations of the angular points of the solid, whose elevation will be formed by joining $O''D'$, $O''A'$, $O''B'$, $O'D'$, $O'A'$, and $O'B'$.

NOTE.—The elevations $O''C'$ and $O'C'$ would be unseen, and are consequently represented as dotted.

Problem 46.

To construct the **projections** *of the* **octahedron** *when it* **lies on its face on the horizontal plane.**

In this case, the solid lies on one of its faces on the horizontal plane. Let *ABC* be the face in the horizontal plane, and *xy* parallel to the axis passing through *E;* then if *AB′* and *CC′* are drawn at right angles to *xy*, *A′C′* is the elevation of the face *ABC.*



Make *A′G′* equal to an edge of the solid ; *G′C′* equal to *B′C′;* draw *C′E′* perpendicular to *A′G′*, and bisected in *O;* join *A′E′* and *G′E′*, thus completing the elevation.

Next, in order to project the plan, we proceed thus.

From *D′*, drop a projector perpendicular to *xy* to meet the lines *BD*, *AG*, drawn parallel to the axis *EC* in *D* and *G*. Join *DE*, *GE ;*

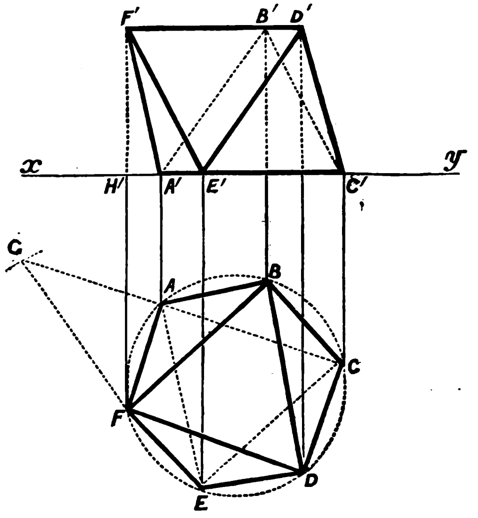
then *EGD* will be the plan of the surface *ED'*. Next join *AE, EB, DC, CG*, to complete the required plan.

Problem 47.

To construct the **plan and elevation** *of an* **octahedron** *when* **resting on one of its faces,** *and when one* **edge** *of this face makes an* **angle of** 15° *with the* **vertical plane.**

Draw *CA* inclined to *xy* at an angle of 15°, and upon it describe an equilateral triangle *ACE*. Describe a circle about the triangle, and in it inscribe a regular hexagon, *ABCDEF*. Join *FD, DB*, and *BF*, which completes the plan of the solid.

Now as the face *ACE* rests on the horizontal plane, the points *A, C, E*, will be projected on *xy* in *A', C', E'*. We now proceed to



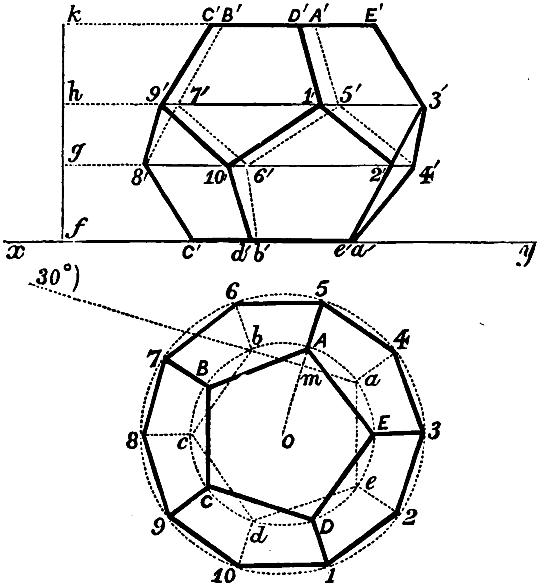
find the height of the given solid. Since *FA* is the projection of a line of which *FB* is the real length, draw *AG* at right angles to *AF*, and with centre *F and radius FB*, describe an arc intersecting *AG* in *G*; then *AG* is the height of the solid.

Make *H'F'* equal to *AG*, and draw a line parallel to the ground line. Cut this line in the points *F', B', D'*, with perpendiculars from *F, B, D*. Then join *F'A', F'E'; B'A', B'C'*; and *D'C', D'E'*; which completes the elevation of the solid.

Problem 48.

To draw the **plan** *and* **elevation** *of a* **dodecahedron,** *when one* **edge** *of its base is* **inclined at an angle of 30°** *to the ground line.*

Take a straight line *ab*, and let it be inclined to the ground line at an angle of 30°, and upon it describe the regular pentagon *abcde*. Describe a circle about *abcde*. Then bisect each side of the pentagon, and draw straight lines through the points of bisection and the opposite angles. These lines will cut the circumference of the circle in *A*, *B*, *C*, *D*, *E*. Join these points, and make *m*5 equal to *mo*, and



from centre *o*, *with radius o*5, describe a circle. By means of the lines already drawn, divide this circle into *ten* equal parts in the points 1, 2, 3, ... 10; and join these points.

To find the elevation of the dodecahedron, project the points *a*, *b*, *c*, *d*, *e*, on the base line. We next find the height of any of the points 2, 4, 6, 8, 10, above the points *a*, *b*, &c. Thus take the point 10; the line *d*10 is the projection of a line whose real length is equal to *ab*, &c. We then find the height of 10 above *d*, as previously shown, and set it off on the perpendicular from *f* to *g*.

We now find the height of any of the points 1, 3, 5, 7, 9, above the points 2, 4, 6, 8, 10. For instance, from 9 to 10 is the projection of a line whose real length is *ab*, &c. Thus the height of 9 above 10 is found as before, and is *gh*.

Lastly, make *hk* equal to *fg*, and through *g*, *h*, *k*, draw lines parallel to *xy*. The points *A*, *B*, *C*, *D*, *E*, will be projected on the line drawn from *k*, the points 2, 4, 6, 8, 10, on the line drawn from *g*, and points 1, 3, 5, 7, 9, will be on the line from *h*.

Then let the points in elevation be joined in the same order as in the plan.

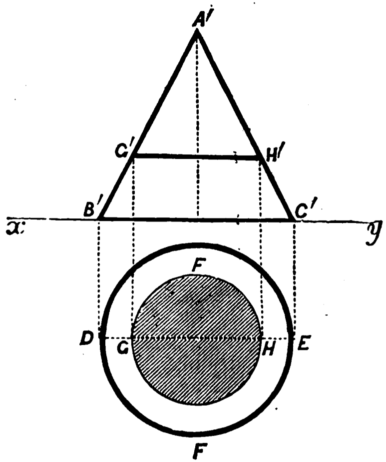
Section VI.—SECTIONS.

In the preceding sections, we have treated on the plans and elevations of the various solids as *wholes.* Now it is often necessary to furnish certain essential details respecting a solid figure, which cannot be obtained by either plans or elevations. To this end, certain *portions* of it are shown, termed *sections;* the consideration of which forms the subject of the present section.

Problem 49.

To draw the **section** *of a* **cone** *when* **cut** *by a* **plane parallel to its base.**

Let the triangle $B'A'C'$ represent the elevation of the given cone,



and the circle DEF its plan, also let $G'H'$ be the **elevation of a section** plane cutting the cone parallel to its base $B'C'$.

Draw *DE*, the diameter of the plan, parallel to *xy*, and from *G'* and *H'* draw projectors cutting *DE* in *G* and *H*. Then the line *GH* is the diameter of the plan of the section. On *GH* construct the circle *GFH*, which will be the plan of the section of the cone cut off by the plane *GH*.

Then let the section be shaded by parallel lines drawn at an angle of 45°, as is usual when sectional drawings are indicated.

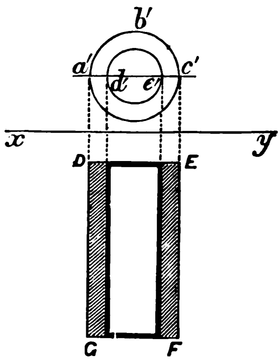
NOTE 1.—If the cone be cut by a plane passing through its axis, the section will be a *triangle*.

NOTE 2.—There are other sections of a cone, such as the ellipse, the hyperbola, and the parabola.

Problem 50.

To draw the section *of a* hollow pipe.

Let *a'b'c'* represent the elevation of the end of a pipe of which *DEFG* is the plan, and let *a'c'* be the elevation of the section plane. Drop projectors at right angles to *xy* from the points where the section plane cuts the figure, as *a'd'e'c'*. In consequence of the section



plane dividing the figure exactly in half, the projections of the lines represented by the points *a'c'* will exactly coincide with the lines in the plan *DG*, *EF*.

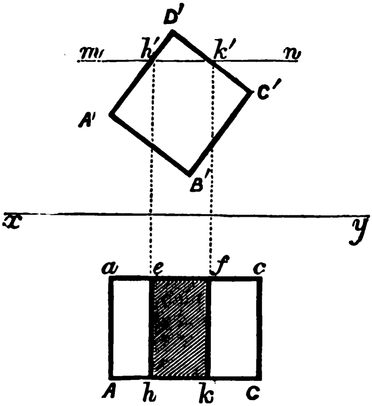
Also, the plan of the semicircle *a'c'* will coincide with the line *GF*. *DE* represents the plan of the other semicircular end of the pipe. It only remains, therefore, to show the *thickness*, which is done by means of the projectors drawn from *d'e'* parallel to *DG* and *EF*.

Problem 51.

To find the **section** *of the* **cube** *given at* **Pr. 9.**

Let *mn* be the section plane, then *mn* represents a horizontal plane, *i.e.*, a plane at right angles to the vertical plane.

It will thus be seen that the cube is cut from the anterior face (from the face turned towards the eye) to the posterior face (the face



turned from the eye). The section produced will be an oblong, having its breadth equal to *h'k'*, and length equal to the edge of the cube.

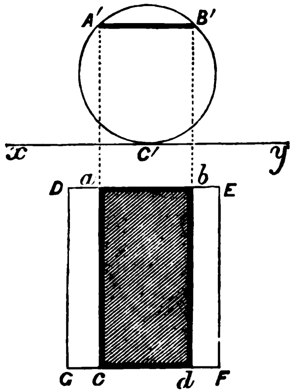
In order to represent the section, find first the plan of the *whole* cube, as in **Pr. 9.** From *h'* and *k'* drop perpendiculars, and we obtain *efkh*, the plan of the required section.

Problem 52.

To draw the **section** *of a* **cylinder** *lying on the ground with its* **ends parallel** *to the* **vertical plane,** *and at* **right angles** *to the* **horizontal plane**; *the* **plane of section being parallel to the axis of the cylinder.**

Let *A'B'C'* represent the elevation of a cylinder lying on the ground, and *DEFG* its plan. Also let *A'B'* represent the elevation of the section plane cutting the cylinder parallel to its axis.

Now as the section plane is parallel to the axis, it is clear that the plan of the section will be a rectangle. In order to determine the



position of this rectangle on the plan, drop perpendiculars from *A′* and *B′*, cutting the ends of the plan in *a* and *c;* and in *b* and *d;* then *abdc* is the plan of the required section.

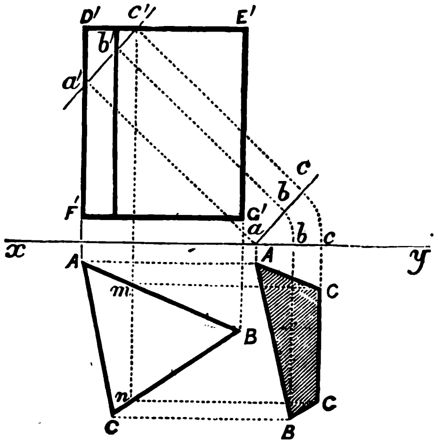
NOTE.—If a cylinder be cut by a plane parallel to its base, the section will be a circle.

Problem 53.

To project the **section** *of a* **triangular prism** *when* **cut** *by an* **oblique plane.**

Let *ABC* be the plan, and *D′E′F′G′* the elevation of a prism which is cut by the section *a′b′c′* oblique to its axis. From points *a′* and *b′*, where the section passes through the angles of the prism, also from point *c′*, where the section passes through the end, draw the projectors *a′a*, *b′b*, *c′c*, to any line *ac* parallel to *a′c′*. Next from *c′* draw the projector *c′n* at right angles to *xy*, and crossing the sides of the prism on the plan at the points *m* and *n*.

From *a* as centre, draw the arcs *bb* and *cc* to *xy;* and from the points *abc*, where these arcs cut *xy*, draw projectors at right angles to *xy*, which intersect the projectors drawn from the points *A*, *C*, *n*, and *m*, on the plan of the oblique section parallel to *xy*. Then the point *A*, where the projector from *A* and *a* intersect, will be one angle of projection, and *B*, where the projectors from *b* and *c* intersect will be another; whilst the other points of projection will be *C* and *C*, where the projector from *c* intersects those from *m* and *n*.



Join the angles of intersection *A*, *B*, *C*, and *C* by straight lines, and the trapezium *ABCC* is the required projection of the oblique section.

Problem 54.

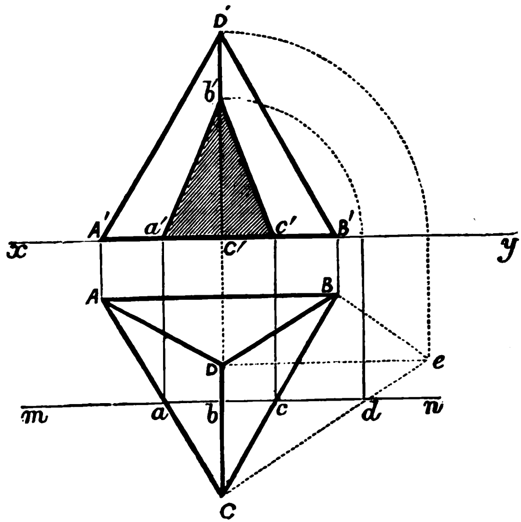
To construct the **sectional elevation** *of a* **tetrahedron** *upon a vertical plane* **parallel to the section plane,** *the trace of which is at right angles to one of the lateral edges of the solid.*

Take *AB*, and upon it describe an equilateral triangle *ABC*. Find *D*, the centre of the triangle, and join *DA*, *DB*, *DC*. We have thus the plan of the tetrahedron when its base is horizontal.

Next, draw *mn*, the trace of the section plane, at right angles to *DC*, and assume this plane to be vertical. We now proceed to find the elevation of the whole solid. In order to do this, we must know the height of *D* above *A*, *B*, *C*. As each of the triangles *DAB*, *DBC*, and *DAC* is equilateral, the real length of *DA*, *DB*, *DC*, is expressed by any of the edges of the base as *AB*, *AC*; consequently from *D* raise a perpendicular to *DC* indefinitely, and with centre *C*, *and radius CB*, describe an arc, cutting this perpendicular in *e*, and join *Ce*.

The points *A*, *C*, *B*, are projected on *xy* in *A'C'B'*; then since *D* is elevated above these points a distance *De;* make *C′D′* equal to this distance, and join *D′A′*, *D′C′*, and *D′B′*.

Now, the section plane cuts *AC*, *CB*, in *a* and *c*, and the projections of these are *a′* and *c′*. It now remains to find the elevation



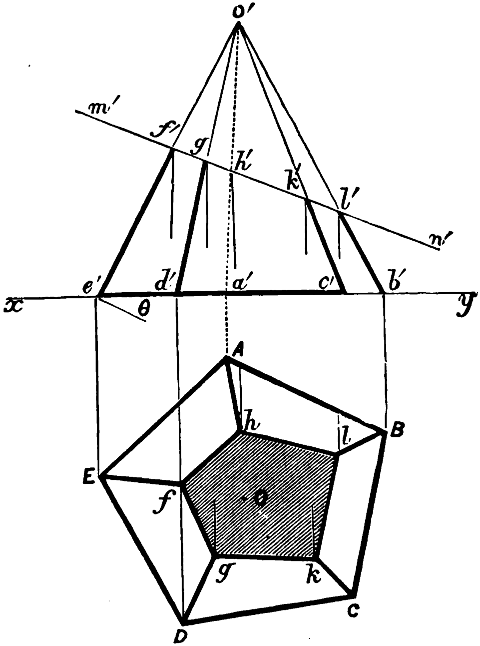
of *b*, the point in which the plane cuts *DC*. It will be seen that *b* is elevated above *C*, a distance *bd;* therefore, make *C′b′* equal to *bd*, and join *b′a′*, *b′c′*, which will complete the sectional elevation.

Problem 55.

To construct the **sectional plan** *and* **elevation** *of a* **pentagonal pyramid** *standing on its* **base** *on the horizontal plane.*

Let *AB* be one edge of the base inclined to *xy* at any angle θ, on *AB* describe the regular pentagon *ABCDE*, and find *O* the centre of the circumscribed circle. Join *OA*, *OB*, *OC*, *OD*, and *OE;* this will be the plan of the required pyramid. Draw *OO′* at right angles to *xy*, and make *O′a′* equal to the perpendicular height of the pyramid; draw *Aa′*, *Bb′*, *Cc′*, *Dd′*, and *Ee′* perpendicular to *xy*, and join *O′a′*, *O′b′*, *O′c′*, *O′d′*, and *O′e′*, which will complete the elevation of the pyramid.

We now proceed to construct the plan of a section made by a plane perpendicular to the vertical plane, its trace *m′n′* making with *xy* the angle θ. Let the trace $m'n'$ meet the elevations $e'O'$, $d'O$, $a'O'$, $c'O'$, and $b'O'$ in the points f', g', h', k', l', respectively; then the plans f, g, h, k, l, will be found by drawing $f'f$, $g'g$, $h'h$, $k'k$, and $l'l$ perpen-



dicular to *xy*; and meeting *EO*, *DO*, *AO*, *CO*, and *BO* in *fghkl* respectively. Then the figure *fgklh* will be the sectional plan required.

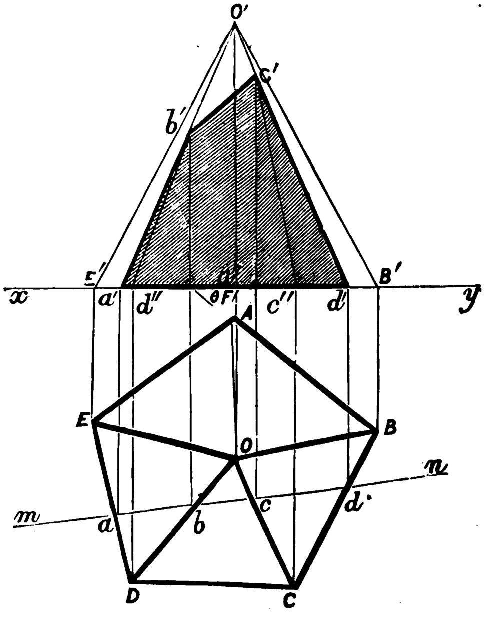
Problem 56.

To construct the **sectional elevation** *and* **plan** *of a* **pentagonal pyramid** *standing on its* **base** *on the horizontal plane.*

Let *AB* be one edge of the base inclined to *xy* at any angle θ; on *AB* describe the regular pentagon *ABCDE*, and find *O* the centre of the circumscribed circle; join *OA*, *OB*, *OC*, *OD*, and *OE*; this will be the *plan* of the required pyramid. Draw *OO'* at right angles to *xy*, and make *O'F'* equal to the perpendicular height of the pyramid;

draw Aa'', BB', Cc'', Dd'', and EE' perpendicular to xy, and join $a''O'$, $B'O'$, $c''O'$, $d''O'$, and $E'O'$, which will complete the elevation of the pyramid.

We next proceed to construct the elevation of a section made by a vertical plane whose trace mn makes with xy an angle θ; let this trace cut ED, DO, OC, and CB in the points a, b, c, d, respectively; then a and d being points in the horizontal plane, their elevations will be in the base line; draw aa' and dd' at right angles to xy; then



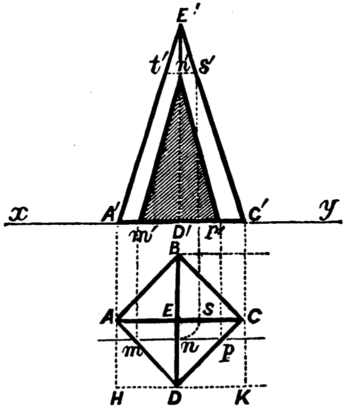
a' and d' will be the elevations of a and d; b and c also are the plans of points in the straight lines whose elevations are $c''O'$ and $d''O'$; if therefore from b and c straight lines be drawn at right angles to xy, and meeting $d''O'$ and $c''O$ in b' and c', then b' and c' will be the elevations corresponding to b and c. Join $a'b'$, $b'c'$, and $c'd'$; then the figure $a'b'c'd'$ will be the sectional elevation required.

Problem 57.

To draw the **vertical projection** *of the* **section** *of a* **square pyramid parallel** *to the* **section plane,** *the trace of the plane, which* *is vertical, being at* **right angles** *to the* **plan** *of one of the lateral edges of the pyramid.*

First, construct the square *ABCD;* join the diagonals *AC*, *BD*, and we have the plan of the pyramid ; the point *E*, where the diagonals cut each other, being its vertex. We now draw *mnp*, the trace of the section plane, at right angles to any of the lateral edges of the pyramid, as *ED.* Draw *xy* parallel to *mnp*. We now find the elevation of the pyramid, which is *E'A'C'*. The points *m* and *p*, being two points in the base, will be projected in *m'* and *p'*, and there remains only to find the elevation of the point *n*.

The vertical projection of *ED* is *E'D'* at right angles to *xy;* and to determine *n* in *E'D'* we want a separate construction. Now, what we require is to find the height of *n* above the horizontal plane,



which height must be set off from *D'* along *D'E'*. This can be done by finding the elevation of *ED* when viewed at right angles to a vertical plane conceived to pass through it, that is, in the direction of *AC*.

Now we have such an elevation of *EC* in *E'C'*, the vertical projection of *EC*. Hence, transfer the point *n* to *s*, by describing an arc, with centre *E*, and radius *En* to cut *EC* in *s*.

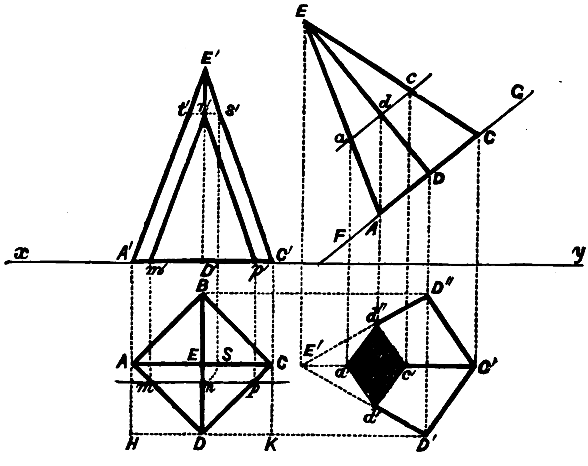
Find *s'*, the elevation of *s* in *E'C'*, and draw *s't'* parallel to *xy;* the point *n'*, in which *t's'* intersects *E'D'* is the vertical projection of *n*. Join *n'm'*, and *n'p'*, and we obtain the required section.

Problem 58.

To draw the **horizontal projection** *of the* **section** *of a* **square pyramid,** *when* **cut** *by a* **plane parallel to its base,** *the plane of the base of the pyramid being inclined at* **40°.**

First construct the square *ABCD;* join the diagonals *AC*, *BD*, and we have the plan of the pyramid; the point *E*, where the diagonals cut each other, being its vertex. Draw *FG*, inclined to *xy*, at an angle of 40°. Then draw *HDK* parallel to *xy*, and set off upon *FG*, *AD*, and *DC* equal respectively to *HD* and *DK*. From *D*, erect *DE* perpendicular to *FG*. Join *EA*, *EC*, and we have an elevation of the pyramid upon the plane *FG*.

Its plan will be found as previously shown. We have now to project the section upon the plan. Draw *adc* parallel to *FG*, to represent the section plane. If the projection of one point in the



section be understood, it will be easily seen how to find the others. For example, the point *c* is a point in the lateral edge *EC*. Now, the plan of *EC* is *E′C′*, hence the plan of *c* must be in *E′C′*, viz. *c′*.

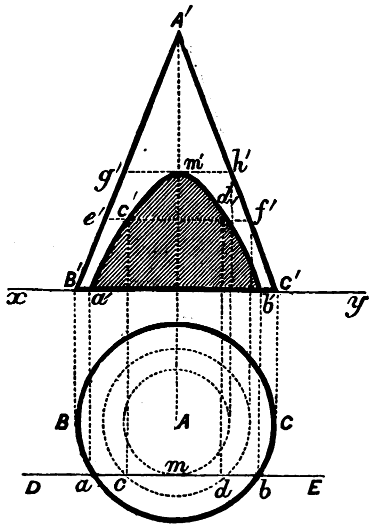
For a similar reason, the plan of *d* must be in *E′D′*, viz. *d′*. The point *d* represents one diagonal of the section, as *D* represents a diagonal of the base of the pyramid. It is clear that the other extremity of this diagonal must be in *E′D″*; and since it must also be in the projector let fall from *d*, it will be in *d″*. Now, the plan of a is a' on $E'C''$; because the plan of EA will be on $E'C'$. Join the points $a'd'c'd''$, and we have the horizontal projection of the required section.

NOTE.—The lines $E'd''$, $E'a'$, and $E'd'$ are dotted in the figure, because this part of the pyramid is supposed to be removed.

Problem 59.

To draw the **sectional elevation** *of a* **right cone,** *when* **cut** *by a* **vertical plane.**

Let ABC and $A'B'C'$ be the plan and elevation of the given cone, DE being the trace of the cutting plane. The plane DE cuts the base of the cone in points a, b, whose elevations are a', b' in B', C'. We have now found two points in the required section. From centre A, describe a circle cutting DE in points c, d. This circle represents



the base of another cone, the elevation of which is $A'e'f'$. Now c and d are two points in the base of the second cone, in the same manner as a and b are two points in the base of the given cone, and their elevations will be upon $e'f'$, just as the elevations of a, b are upon $B'C'$. We thus get $c'd'$, two more points in the required section. Further, from centre A describe a circle tangential to DE, and touching it at *m*. This circle represents the base of a third cone, whose elevation is $A'g'h'$. The elevation of *m* is m', which gives the height of the section. Through points a', c', m', d', b', describe a curve as shown in the diagram, and thus obtain the section required.

NOTE.—The curve which shows the outline of the section is a *hyperbola*.

Problem 60.

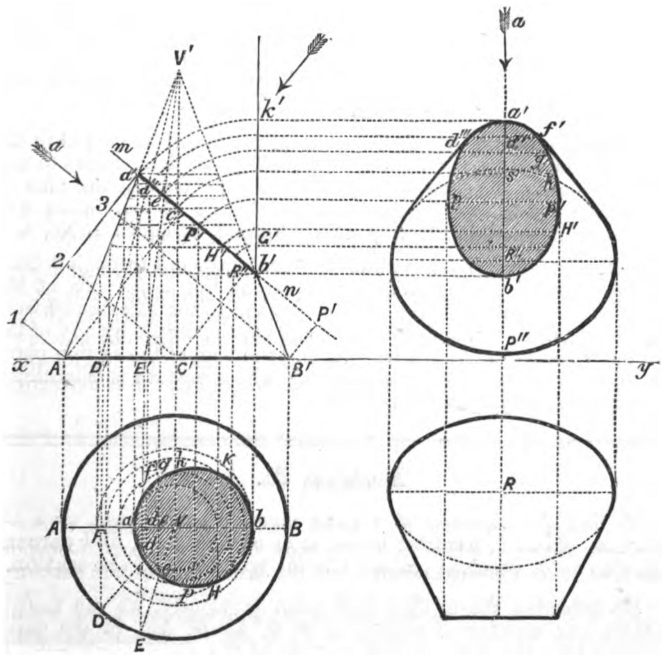
To find the projection of a **cone** *standing on its* **base,** *when the* **section plane** *is* **perpendicular** *to the* **vertical plane,** *and making an* **angle with the horizontal plane**; *also a projection of the cone, showing the* **true form of the section.**

First, through the vertex of the cone draw a line $V'E'$ to *any* point within the base $A'B'$; this line is to be considered as the vertical projection of a generatrix of the cone, and the point e' where it cuts the line *mn*, is the projection of that point on the surface of the solid, where the cutting plane actually passes through the generatrix $E'V'$. The point e' may be projected upon the plan by letting fall a perpendicular from E', cutting the circumference of the base in *E*, and joining *EV*; then another perpendicular let fall from e' will intersect *EV* in a point *e*, which will be the horizontal projection of a point in the required curve. By drawing another line, *e.g.*, $V'D'$, and projecting its point of intersection d' with the cutting plane, to *d*, a second point in the curve is obtained; and so on for any number of points required.

The exterior generatrices $A'V'$ and $B'V'$, being both projected upon the line *AB*, the extreme limits of the curve sought will be at the points *a* and *b*, on that line, which are the projections of the points of intersection a' and b', of the cutting plane with the outlines of the cone. And, as the line *ab* will clearly divide the curve symmetrically into two equal parts, the points *f*, *g*, *h*, &c., will be readily obtained by setting off above that line, and on their respective perpendiculars, the distances, *dd*, *ee*, &c. A sufficient number of points having thus been determined, the curve drawn through them (which will be found to be an ellipse) will be the outline of the required section.

This curve may be obtained by another, and perhaps simpler method, depending on the principle that all sections of a cone by planes parallel to the base are circles. Thus, let line $F'G'$ represent a cutting plane; the section which it makes with the cone will be denoted, on the horizontal projection, by a circle drawn from the centre *V* with a radius equal to half the line $F'G'$; and by projecting the point of intersection H', of the horizontal and oblique planes, by a perpendicular $H'H$, and noting where this line cuts the circle above referred to, we obtain the points *H* and *K* in the curve required. Similarly, any number of additional points may be found.

Secondly, let the cutting plane *mn* be conceived to turn upon the point b', so as to coincide with the vertical line $b'k'$, and let $b'k'$ be transferred to $a'b'$, which will represent as before the extreme limits of the required curve. Now, taking any point, such as d', it is clear that in this new position of the cutting plane, it will be represented by d'', and that if we make the further supposition that the cutting plane were turned upon $a'b'$, as an axis, till it should be



parallel to the vertical plane, the point which had been projected at d'' would then have described round $a'b'$ an arc of a circle whose radius is the distance *dd*, No. 2. This distance, therefore, being set off at d''' and f', on each side of $a'b'$, gives two points in the required curve. By a similar mode of operation, *any* number of points may be obtained, through which, if we draw a curve, it will be an ellipse, of the true form and dimensions of the section. Or, having found the axes, major and minor, the ellipse may be constructed by any of the methods referred to in *Plane* Geometry.

Further, to complete the projection of the cone when seen in this new position; if we look at No. 1 in the direction of the arrow *k'*, which is at right angles to the section plane, it will be seen that by using this plane as the plane of projection, all the points necessary for the figure may be obtained upon it. For example, the base of the cone *A'B'*, whose true form is a circle, being at an angle with this plane, its projection will be an ellipse. From *B'*, *C'*, and *A'*, draw projectors to meet the line *mn* in *P'*, &c. From *P'*, take the distance *P'b'*, and set it off from *b'* to *P''*. Now the point *R'* will represent the centre of the base. Take *P'R'*, and set it off from *P''* to *R''*, and the same distance from *R''* to *S'*; then *P''* and *S'* will be the projections of the *minor* axis of the ellipse. Through *R'*, draw a line parallel to *xy* and make it equal to *A'B'* for the *major* axis. An ellipse described about these axes is the base of the cone, and lines drawn tangential to the two ellipses will complete the figure.

Lastly, to obtain the *plan* of this figure, it will be viewed in the direction of arrow *a*. From *a'*, draw a line at right angles to *mn*. If we conceive this line to represent an edge view of the plan of projection, projectors drawn from the points of the figure will represent on this line the projections required, as shown in No. 1.

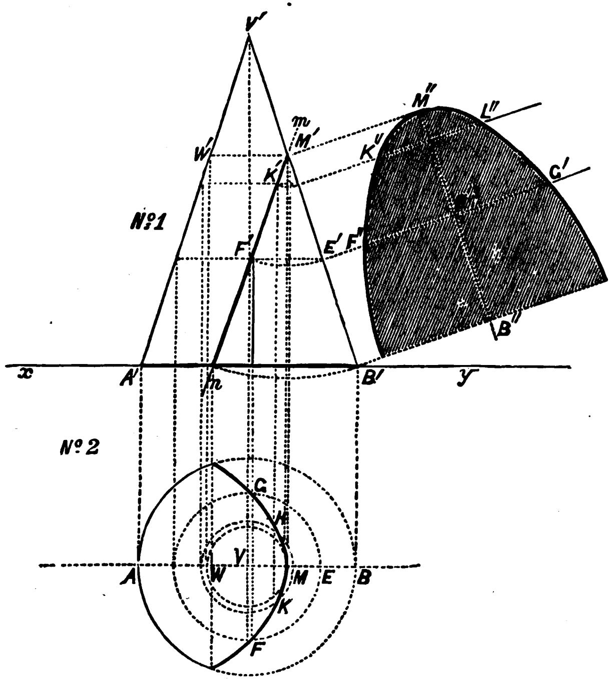
Take any point, *R*, on the line *a'P''* produced, and through it draw a line parallel to *xy*; this will represent the *major* axis of the ellipse as seen in the figure. To obtain the *minor* axis, set off from *R*, the distances 1 2, and 2 3, and complete the ellipse by any of the ordinary methods. It will be seen that the plan of the section plane in this position becomes a straight line, whose breadth is determined by dropping projectors from *p* and *p'*.

Problem 61.

To find the projection of a **cone** *standing on its* **base,** *when the* **section plane** *is* **parallel** *to one* **side of the cone,** *and* **perpendicular** *to the* **vertical plane**; *also the* **true form of the section.**

By following the method laid down previously, we can readily obtain any number of points, as *F*, *G*, *K*, *W*, &c., in the curve representing the horizontal projection of the section specified. It must be remarked that the horizontal plane passing through *M'* gives only one point *M* (which is the vertex of the curve required), because the circle which denotes the section that it makes with the cone is a tangent to the given plane.

In order to determine the actual outline of this curve, let us suppose the plane *mn* to turn, as upon a pivot at *M'*, until it has assumed the position *M'B'*, and transfer *M'B'* parallel to itself, to *M''B''*. The point *F'* will thus have first described the arc *F'E'* till it reaches the point *E'*, which is then projected to *E''*; so that, if we conceive the given plane, now represented by $M''B''$, to turn upon that line as an axis, until it assumes a position parallel to the vertical plane, we shall find that the point E', which is distant from the



axis $M''B''$ by the distance FV, No. 2, will now be projected to F'', No. 1. The same distance FV, set off on the other side of the axis $M''B''$, gives another point G' in the curve required, which is that called the *parabola.*

Section VII.

PENETRATIONS OF SOLIDS.

The preceding sections being thoroughly understood, the following problems will present no difficulty to the student. All three relate to cylinders, and are of an elementary character.

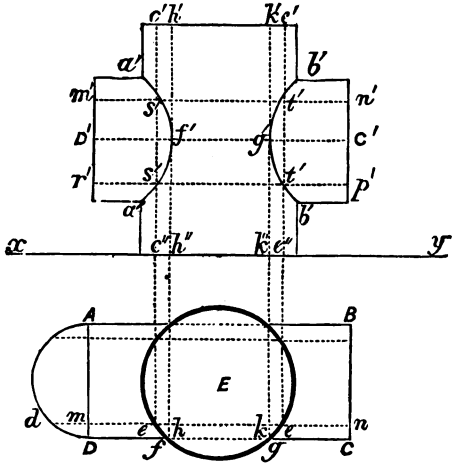
Problem 62.

To draw the **curve of penetration** *of* **two right cylinders,** *whose* **axes** *are at* **right angles** *to each other.*

Let *ABCD* be the plan of the horizontal cylinder, and the circle *E* the plan of the vertical cylinder. Find the vertical projection of the two cylinders. Thus we get four points in the curve of penetration required, viz., a', a', b', b'. Draw *mn* parallel to *DC*, and let it represent a section plane at right angles to the horizontal plane, cutting both cylinders. Now, this plane cuts the vertical cylinder in a rectangle whose elevation is $c'c''e''e'$, which is found by raising perpendiculars from the points *c* and *e*, where the plane *mn* cuts the circle *E*.

The plane *mn* also cuts the horizontal cylinder in a rectangle, the elevation of which is $m'n'p'r'$. It is found thus;—On *DA* a semicircle is described, representing half the base or end of the cylinder *ABCD*. Produce *mn* to meet the circumference of the semicircle in *d*. The ordinate *md* represents half the width of the rectangle, which is the section of the horizontal cylinder by the plane *mn*. Hence, from D' ($D'C'$ is the elevation of *DC*) set off $D'm'$, $D'r'$, each equal to *md*, and through $m'r'$ draw $m'n'$, $r'p'$ parallel to $D'C'$. The rectangle $d'c''e''e'$ intersects the rectangle $m'r'p'n'$ in the points $s's'$ and $t't'$. We have thus found two more points in the required curve.

Lastly, taking DC as another section plane, this plane cuts the vertical cylinder in a rectangle, whose elevation is $h'h''k''k'$, found by erecting perpendiculars from the points h, k, where DC intersects the circle. Now, this rectangle intersects $D'C'$ in f' and g', the elevations of f and g points in DC. The curve described through $a's'f's'a$

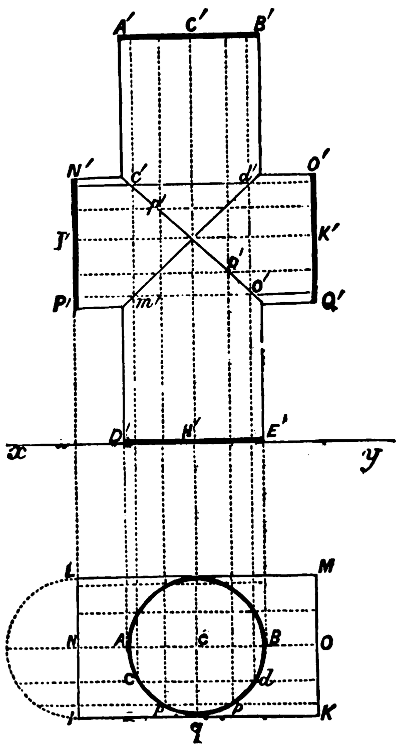


and $b't'g't'b'$ is that in which the horizontal cylinder $ABCD$ intersects the vertical cylinder E.

Problem 63.

To draw the **curve of penetration** *of* **two right cylinders,** *whose* **diameters** *are* **equal,** *and whose* **axes** *are at* **right angles** *to each other.*

From the following remarks it will be seen in what respects the present problem differs from the foregoing. The diameters of the cylinders being equal (**Pr. 63**), the curves of penetration are projected vertically in straight lines perpendicular to each other. For, if we proceed to apply the method before given, we shall soon discover that the various points in these curves are situated in two planes at right angles to each other, and to the vertical plane, the sections formed by them being, in fact, ellipses equal and similar to each other. It is not necessary to enter into any details in illustration of this case, other than to call attention to the figure, where the

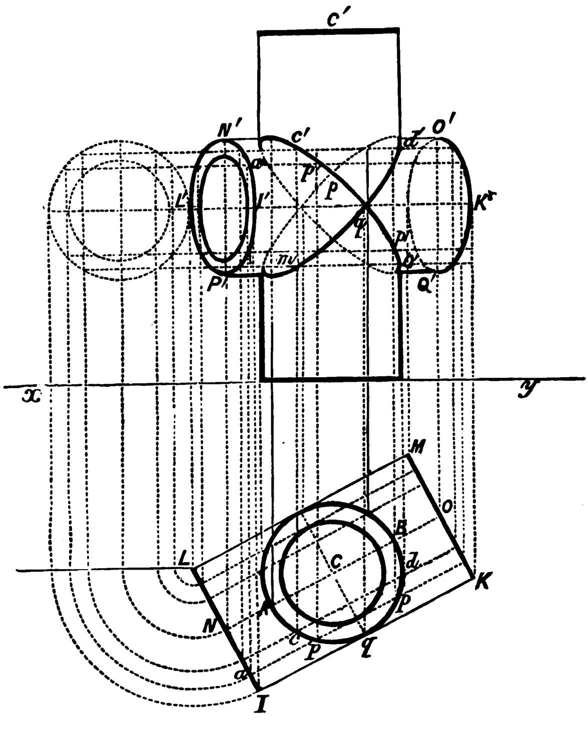


projections of some of the points are indicated, both in elevation and plan, *by the same letters of reference.*

Problem 64.

To draw the **curve of penetration** *of* **two right cylinders,** *whose* **diameters** *are* **equal,** *and whose* **axes** *are at* **right angles** *to each other,* **one of the cylinders being inclined to the vertical plane.**

The two preceding figures being drawn (**Prs. 62** and **63**), we may easily find the projection *c*, of any point such as *c′*, by observing that it must be situated in the perpendicular *c′c ;* and that, since the distance of this point (projected at c' in **Pr. 63**) from the horizontal plane remains unaltered, it must also be in the horizontal line $c'd'$. Upon these principles all the points indicated by *literal* references in the present problem are determined; the curves of penetration resulting therefrom intersecting each other at two points projected



upon the axial line $L'K'$, of which that marked q' alone is seen. The ends of the horizontal cylinder are represented by ellipses, the construction of which will also be clear on referring to the figure, and they do not require any further consideration.

MISCELLANEOUS EXERCISES IN PRACTICAL PLANE GEOMETRY.

(A.)

1. Draw a straight line 8 inches long, and divide it into 16 equal parts by continual bisection.

2. Make any line *AB*, 2 inches long, divide it into four equal parts, and at each end and point of division erect a perpendicular $1\frac{1}{2}$ inch in height.

3. Draw any vertical line *AB*, 4 inches in length. From the upper extremity, draw a line *AC*, 3 inches in length, and at right angles to it. Then bisect each of these lines.

4. Construct any triangle, then draw a line perpendicular to the base, and passing through the apex.

5. Construct a scale to represent 20 miles, taking $\frac{1}{8}$th of an inch to the mile.

6. Draw a horizontal line *AB*, 3 inches long. From *B*, drop a line *BC* at right angles to *AB*, 2 inches in length. Then trisect the right angle, and lastly, bisect each of the trisections.

7. Construct an equilateral triangle, and on its three sides respectively construct a square, a hexagon, and a rhombus having an angle of 45°.

8. Draw a line to represent 60°, as marked on the side of a map, on a scale of 10° to half an inch.

9. Draw any two parallel lines *AB* and *CD* at any distance apart. Find a point *E*, which shall be equidistant from these lines.

10. Draw a circle of $1\frac{3}{4}$ inches radius. Divide it into 6 equal parts, and at each of the points of division, draw a line tangent to the circle.

11. Through any given point *A* within a circle, whose radius is $1\frac{1}{2}$ inch, draw the longest possible chord.

12. There is a stick leaning against a vertical wall, and making an angle of 60° with the ground. Required the angle which the stick makes with the wall.

(B.)

13. Draw a square of $3\frac{1}{4}$ inches side, and inscribe in it four equal circles, each touching two others, and two sides of the square.

14. Divide the area of any given circle into six equal sectors, by lines drawn from the centre.

15. Draw a vertical line AB, 3 inches in length. On AB, construct a triangle, having an angle of 50° at A, and an angle of 40° at B. Then state how many degrees the remaining angle contains.

16. Produce a line AB, 3 inches in length, to a point C, so that $BC : AB :: 3 : 5$.

17. Divide a circle into three proportional areas by means of concentric circles, so that the area of the outside circle is three times that of the inside one, and the middle area twice that of the inside one.

18. Construct a triangle having sides respectively of 4 inches, 3 inches, and $2\frac{1}{2}$ inches. On the 4 inches side, mark off any four *irregular* divisions, then divide the $2\frac{1}{2}$ inches side proportionately to the divisions on the 4 inches side.

19. Line AB is 3 inches in length, CD is 2 inches, and DE is $1\frac{1}{2}$ inch. Find a line, FG, so that $CD : AB :: FG : DE$.

20. Draw a tangent touching an arc in any given point A, without using the centre.

21. Construct a right-angled triangle, making the hypotenuse twice the length of the base.

22. AB is the mean proportional between two lines 3 inches and 1·5 inch. Find its length.

23. In a given circle whose diameter is 2 inches, inscribe a regular pentagon in two different ways.

24. Draw a vertical line AB, 4 inches in length. Let this line be the altitude of an equilateral triangle. Construct it.

(C.)

25. Draw an equilateral triangle, whose perimeter shall be equal to a square of 1·5 inch.

26. Show the position of a wheat sheaf situated exactly in the middle of a corn-field, which is bounded by six equal hedges.

27. Draw a sector of 3 inches radius, and having an angle of 150°.

28. Construct a right-angled triangle, whose base is 2 inches, the acute angles being in the ratio of 2 : 1.

29. Prove by illustrations that the angles made by straight lines drawn from the centre of any polygon to the angular points, are together equal to four right angles.

30. Draw a horizontal line *AB*, 2 inches in length. On *AB*, as base, construct a triangle, having sides of 2 inches and 3 inches, and find the altitude of the triangle.

31. Determine an equilateral triangle, equal in area to the sum of two squares, having their sides 1 inch and 2 inches respectively.

32. Construct a regular polygon whose side *AB* is the chord of an arc of 45°.

33. Draw the plan of a rectangular field, 400 yards by 250 yards. Mark a point *O*, which shall be exactly the centre of the field. [*Scale*—1 *inch to* 100 *yards.*]

34. Draw an equilateral triangle of 2 inches side, and a square equal to it in area.

35. Construct a regular polygon on any given line *AB*, having the distance from either of its extremities to the centre equal to the side *AB*.

36. Draw a horizontal line *AB*, 2¼ inches in length. Let this line be the altitude of an isosceles triangle. The altitude makes an angle of 15° with one of the sides of the triangle. Construct the triangle.

(D.)

37. Construct a square, an equilateral triangle, and a hexagon. Determine by a square the area of the three figures added together.

38. Inscribe in any given circle a triangle that shall cut off equal segments.

39. Draw a rhomboid, letting the shorter side be half the length of the longer side, and one of the angles to contain 60°. Find the centre of the rhomboid, and from it draw a line perpendicular to one of the longer sides.

40. Draw three circles of 1, 1½, and 2 inches radii, so that each circle touches the other two.

41. In a given circle 2½ inches in diameter, inscribe seven equal circles, six of which shall touch the given circle and a central one.

42. Draw a circle of 1 inch radius. Outside of this circle, find the positions of two points, *A* and *B*, which are to be respectively 1½ inch and ¾ inch from the circle. *A* and *B* are also to be 3 inches from each other. Draw a couple of tangents to the circle, from each of these points.

43. Draw two lines at an angle of 40°. Then draw two circles, each touching the lines and one another, the radius of the smaller one to be 2 inches.

44. In any given square sufficiently large, inscribe a triangle having its two sides equal and its base one inch long.

45. Describe an arc of a circle, and show how its centre may be ascertained if it were not already marked.

46. In any given square inscribe a regular polygon that shall cut off four equal corners of the square.

47. Describe a circle of 1½ inch radius. Mark any point *A*, on the circumference. Inscribe another circle of 1 inch radius within the first circle, and which shall just touch the large circle at point *A*, tangentially. Then describe another circle of 1¼ inch diameter, which shall be outside of the large circle, and also just touch point *A*, tangentially.

48. Draw a tangent touching the curve of an ellipse at any given point *A*, and also a line perpendicular to the curve, from that point.

(E.)

49. Draw the plan of a triangular piece of wood, having sides respectively 3 feet, 2¼ feet, and 1½ foot. It is required to cut the largest possible circle out of this piece of wood. Show how large the circle would be. [*Scale*—1 *inch to the foot.*]

50. Draw a rectangle having sides of 3 inches and 2⅛ inches, and in it inscribe an ellipse.

51. Draw a line *AB*, an inch long. From *B*, draw line *BC*, 2 inches long, and making an angle of 30° with *AB*. These lines are adjacent chords of a circle ; describe the circle.

52. Within an octagon whose base is 1½ inch, inscribe a similar concentric octagon whose base is ¾ inch.

53. There is a thin piece of metal in the shape of an isosceles triangle. Its base is 3 inches long, and the angle at the apex is a right angle. Show how to cut this into four smaller equal triangles, all similar in shape to the large triangle.

54. About a given heptagon whose base is 1·5 inch, describe a similar heptagon whose base is 2 inches.

55. Draw a vertical line *AB*, 2 inches long. Let this be a diameter of a square. Construct the square, and then divide it into nine smaller squares, equal to each other.

56. About a given circle describe a triangle having angles of 20°, 60°, and 75° respectively.

57. On a given line *AB*, 2 inches long, construct a regular hexagon, by the use of the set-square of 60°.

58. Draw three lines 1½ inch, 2 inches, and 2½ inches respectively, and find their fourth proportional.

59. Draw a rectangle of 4 inches and 2¼ inches sides. Within it inscribe an ellipse, which shall touch the centre of each side tangentially.

60. Find the length of *AB*, the mean proportional between two lines, 1½ inch and 3 inches long.

(F.)

61. Draw a square, and by means of parallels to its sides—at a distance of ½ inch—construct another square.

62. Draw a scale to represent 15 feet, on a scale of ⅛th of an inch to an inch.

63. Let a line *AB* (6 inches) represent the plan of one side of a street. Show the plan of the opposite side, which is to be parallel to *AB*. The street is to be 15 feet wide. [*Scale*—1 *inch to* 10 *feet.*]

64. Draw a line 4·5 inches in length, and at one extremity erect a perpendicular 1·5 inch long. From the top of the perpendicular draw a line making an angle of 30° with the given line.

65. Draw the plan of a circular race-course 1 mile in diameter, so that three gates, *A*, *B*, and *C*, shall fall within the path. [*Scale*—1 *inch to a mile.*]

66. Draw a horizontal line *AB*, 2 inches in length. Let this be the base of a triangle having one side $\frac{4}{7}$ the length of *AB*, and the other side $\frac{6}{7}$ of *AB*. Construct the triangle.

67. Draw a square of 2·5 inches side, and inscribe another within it, having each of its corners in the sides of the first, and at 1 inch from its angular points. Describe the circles circumscribing these two squares.

68. From a given point *A*, 1½ inch outside the circumference of a given circle whose diameter is 2 inches, draw a tangent to the circle.

69. An isosceles triangle has one of the angles at its base equal to 36° 14′ 23″, what will be the vertical angle?

70. A line AB is 3·5 inches long. Divide it in the point C, so that $AB : BC :: 7 : 4$.

71. Describe a circle which shall pass through three consecutive angles of any regular nonagon.

72. The diagonal of a rectangular drawing-board is $2\frac{1}{4}$ feet. One of the sides of the board makes an angle of 30° with one end of the diagonal. Draw the plan of the board. [*Scale*—1 *inch to the foot.*]

(G.)

73. Divide a line 6 inches in length in extreme and mean proportion, and prove by construction that the greater segment is a mean proportional between the whole line and the less segment.

74. Inscribe in any given circle, that polygon whose angles at the centre are each 60°.

75. Draw any irregular polygonal figure, say an irregular pentagon. Let this represent the plan of a field, and draw another plan similar and equal to it.

76. Construct a triangle, two of its sides being 3 inches and 2 inches respectively, and the included angle 50°.

77. About any given circle, describe a regular pentagon, whose sides are parallel to an inscribed pentagon.

78. Construct a rectangle, making one of the sides $\frac{5}{8}$ of the length of the adjacent side.

79. Place two equal lines, 1 inch long, at any angle of 135°. Consider them as two sides of a polygon, and complete the figure.

80. In any scalene triangle whose base is 2 inches, inscribe a rectangle having its base 1·5 inch.

81. Describe a circle of 2 inches diameter. Inscribe within it an irregular polygon, having angles at the centre, equal respectively to 45°, 60°, 30°, 105°, and 120°.

82. Construct a triangle ABC, having its angles 50°, 60°, and 70°, and circumscribing a circle of 1 inch radius.

83. Describe a *tre-foil* and *quatre-foil* having adjacent diameters of $\frac{5}{8}$ of an inch.

84. State how many degrees are contained in the angles of each of the following regular polygons—a hexagon, an octagon, and a do-decagon.

(H.)

85. Draw a pentagon, having its side $1\frac{1}{2}$ inch in length. Divide it into five isosceles triangles, by drawing lines from its centre to the angular points, and inscribe a circle in each.

86. Construct a *quatre-foil* and *cinque-foil*, having tangential arcs, the radius of which is $\frac{3}{4}$ inch.

87. On a given line AB, 1 inch in length, construct a regular octagon, with a set-square of 45°.

88. Bisect a triangle, having its sides 4·5, 5, and 5·5 inches, by a line drawn perpendicular to the longest side.

89. Construct an ellipse in two different ways, the transverse diameter AB, and conjugate diameter CD, being given.

90. Draw a sector of 3 inches radius, and containing an angle of 120°. Divide it into six smaller sectors, all equal to one another.

91. Make any irregular figure of six sides, and construct an equilateral triangle equal in area.

92. The base of a scalene triangle is $\frac{3}{4}$ inch. Its angles are respectively 40°, 60°, and 80°. Describe a similar triangle, whose base is $1\frac{1}{4}$ inch.

93. Draw an oblique line AB, 4 inches in length, and let it represent the major axis of an ellipse. Then draw the minor axis a length of $1\frac{1}{2}$ inch. Describe the curve of the ellipse, by means of intersecting arcs, and draw a tangent through any point C in the curve.

94. Describe two circles equal to the sum and difference respectively of two other circles of 1·5 inch and 3 inches diameter.

95. From a circle, whose radius is $1\frac{5}{8}$ inch, cut off a segment which shall contain an angle of 50°.

96. Find the mean proportional to two lines AB and CD, being 3 inches and 2 inches respectively in length.

(I.)

97. Construct a pentagon having a diagonal 4 inches in length, and a square equal to it in area.

98. The perimeter of a triangle is 6 inches. Construct it so that its sides are in the proportion of 2, 3, and 4.

99. About a circle of 3 inches in diameter, construct a triangle having an angle of 30° and another angle of 45°.

100. The centres of two circles are 2·5 inches apart, having their radii respectively $\frac{3}{4}$ inch and $\frac{1}{2}$ inch. Draw the four lines touching both circles.

101. Draw an equilateral triangle having a base of $2\frac{1}{2}$ inches, and construct a rectangle equal to it in area.

102. Within a square of 3 inches side, inscribe the largest possible equilateral triangle.

103. A line 17·5 yards long is represented on a certain drawing by 3·5″. Construct the scale to show yards.

104. Construct a rhombus having a base of 4 inches, and two angles of 45°, and make a triangle of equal area having one angle of 70°.

105. From any point *C* in the circumference of a circle of $1\frac{1}{2}$ inch radius, draw a chord which will cut off a segment containing an angle of 30°.

106. Determine $\frac{1}{96}$ of an inch by diagonal division.

107. Draw a rhomboid on a base of 1·5 inch, and construct an isosceles triangle of equal area.

108. On a given line *AB*, $2\frac{1}{2}$ inches long, construct an equilateral triangle. On the same line construct a scalene triangle, of the same area as the equilateral triangle, and having one of its angles equal to 20°.

(K.)

109. Construct an equilateral triangle equal in area to a square, the base of which is $1\frac{1}{2}$ inch.

110. Construct a square of $1\frac{1}{2}$ inch side. Then on one of the sides construct an isosceles triangle equal to the square in area.

111. Construct a triangle equal in area to two similar triangles, the bases of which are respectively 1 inch and $1\frac{1}{2}$ inch.

112. Draw any irregular seven-sided rectilineal figure. On a given line *AB*, $2\frac{1}{2}$ inches long, construct a rectangle equal in area to the irregular figure.

113. Construct a triangle equal in area to the difference between two similar triangles, the bases of which are respectively $2\frac{1}{2}$ inches and $1\frac{1}{2}$ inch.

114. Draw a regular pentagon of $1\frac{1}{2}$ inch sides. Then construct the following figures, each having $\frac{2}{3}$ the area of the pentagon;—viz. a square, a right-angled triangle, and a rhomboid, the latter containing acute angles of 45°.

115. Describe a circle having a radius of 1½ inch, and construct a rhomboid equal to it in area, and having an angle of 45° at the base.

116. Within an equilateral triangle of 2 inches sides, insert a trefoil of tangential arcs of circles.

117. Draw a circle having a radius of ¾ inch, and a line *AB* at any distance from it. Draw a circle which shall touch both the given line and circle.

118. Draw a trapezium having adjacent pairs of sides equal, and respectively 2½ and 3½ inches in length. Within the trapezium, inscribe a circle and a square.

119. Two lines, *AB* and *CD*, converge towards each other. Show how the angle at which they meet may be bisected when it is inaccessible.

120. Draw a sector containing an angle of 120°, and having radii of 1½ inch. Inscribe a circle within this sector, which shall touch the arc and the radii, tangentially.

END OF PLANE GEOMETRY EXERCISES.

MISCELLANEOUS EXERCISES IN PRACTICAL SOLID GEOMETRY.

(A.)

1. Project a line 3 inches long when parallel to the horizontal plane, and at right angles to the vertical plane, its height above the ground being 2 inches, and distance from the vertical plane 2 inches.

2. *A* and *B*, 3 inches apart, are the *plans* of two points, of which *A′* is 2 inches, and *B′* 3·5 inches, above the paper. What is the length and inclination of to the paper, of the line *A′B′*?

3. Draw the plan and elevation of a black-board 4 feet square, suspended 3 feet above the floor of a schoolroom to which it is parallel and against the wall. [*Scale,* 2 *feet to an inch.*]

4. Draw the *plan* of a line 4 inches long when inclined at 45°, and an elevation of it on any vertical plane not parallel to the line.

5. Project a piece of straight wire 3 feet long, which is fixed in the wall at right angles to it, and 6 feet above the ground to which it is parallel. [*Scale,* 2 *feet to an inch.*]

6. The plan of a line is 1·5″ long, and its elevation is 3″. The *projectors* of its extremities are 1″ apart, measured along *xy*. What is its true length and inclination?

7. Project a square prism, one end of which rests on the horizontal plane, and one of its upright faces is parallel to the vertical plane. The height of the long edges is 8′, and of the end edges 4′. [*Scale,* $\frac{1}{8}$″ *to the foot.*]

8. Draw the plan and elevation of a point *A*, which is situated above the horizontal plane, 3″ behind the vertical plane, and 3″ distant from *xy*.

9. Project a triangular prism resting on one of its ends, and having one of its faces parallel to the vertical plane; its height being 10′, and the breadth of each of its triangular edges 4′. [*Scale,* $\frac{1}{8}$ *inch to the foot.*]

10. A line *AB*, 4″ long, is inclined 50° to the horizontal plane. Draw its projections when its plan makes an angle of 35° with *xy*.

11. Project a hexagonal prism resting on the floor of a room having one of its faces inclined at an angle of 45° to the vertical plane, the height of the prism being 8′ and the width of each face 4′. [*Scale,* 6 *feet to an inch.*]

12. Draw both plan and elevation of a cube of 3″ edge, when its base is horizontal, and ·5″ above the paper; its horizontal edges making angles of 35° with the vertical plane.

(B.)

13. A globe, 10″ in diameter, stands on a square table, the edge of which touches the wall of a room. Draw its plan and elevation on a scale of 1′ to 1″.

14. Draw plan and elevation of a square prism *of any size,* when its long edges are horizontal, and one of its faces makes an angle of 35° with the paper.

15. Project a cylinder resting on the horizontal plane on one of its ends; its height is 6′, and the diameter of its base 3′. [*Scale,* $\frac{1}{4}$′ *to the foot.*]

16. Draw both plan and elevation of a tetrahedron of 2″ edge, when its axis is vertical.

17. Project a cone resting on the horizontal plane, its vertical height being 6′ and the diameter of its base 3′. [*Scale,* $\frac{1}{4}$″ *to the foot.*]

18. A cone, base 1·5″ radius, 3″ high, is cut by a plane at 70° with the axis; the centre of the section being 2″ above the base. Show the plan of the cut.

19. Project a tetrahedron of any size, having one of its faces resting on the horizontal plane, and one side inclined at an angle of 40°.

20. Draw plan and elevation of a square pyramid, base 1″ side, height 4″ when one of its long edges is inclined 20° to the paper.

21. Project a pentagonal prism which stands on the floor on one of its ends, having a hidden face parallel to the horizontal plane. Height 5 inches, and width of face $1\frac{1}{2}$ inch.

22. A hexagonal prism, base 1″ edge, and 3″ long, has its axis horizontal, one of its faces being inclined 15° to the paper. Draw both plan and elevation, and a second elevation upon a vertical plane, making an angle of 45° with the plan of the axis.

23. Draw the elevation and plan of a cone, the height of the elevation being 3 inches, and the width of the base 1·5 inch.

24. A pyramid having for its *base* a square 3″ side, and its axis 3·5″ long, rests with one *face* on the horizontal plane. Draw its plan, and a sectional elevation on a vertical plane represented by a line bisecting the plan of the axis, and making an angle of 60° with it.

(C.)

25. *AB* is the elevation of a line 4 feet long, which is parallel to the horizontal plane, but inclined to the vertical plane. Project its *plan* and determine the angle at which it is inclined. [*Scale,* $\frac{1}{4}''$ *to the foot.*]

26. The horizontal and vertical traces of a certain oblique plane make angles of 40° and 80° respectively with *xy*. Assume any point above the base line as the elevation of a point contained by this plane, and determine its plan.

27. *AB* is the plan of a line 4 feet long, which is parallel to the vertical plane, but inclined to the horizontal plane. Project its *elevation,* and determine the size of the angle of inclination. [*Scale* $\frac{1}{4}''$ *to the foot.*]

28. Draw a line parallel to *xy*, at a distance of 1·5″ from it. Consider this as the horizontal trace of a certain plane inclined 40° to the horizontal plane, and determine the vertical trace.

29. Project an equilateral triangle, its surface being inclined at an angle of 45° to the vertical plane, with its base parallel to the horizontal plane.

30. Draw two parallel planes, inclined 50° to the horizontal plane, and 1″ apart; their horizontal traces to make angles of 40° with *xy*.

31. Project a regular hexagon at an angle of 40° to the vertical plane, its axis being parallel to the horizontal plane.

32. Draw the traces of a plane inclined 75° to the horizontal plane, and 35° to the vertical plane.

33. Project a hexagonal prism, resting on one of its solid angles; its axis being inclined to the horizontal plane at an angle of 60°, but parallel to the vertical plane. Its height is 8′, and the width of each of its faces 4′. [*Scale* $\frac{1}{8}''$ *to the foot.*]

34. A square has its surface inclined 45°, neither of its sides being horizontal. Draw plan and elevation.

35. A square prism, base 2″ by 4″ long, has one of its rectangular faces inclined 40°, the diagonal of that face being horizontal. Draw plan and elevation.

36. The axis of a square pyramid, base 1·5″ side, 4″ long, is inclined 60°, one edge of the base being horizontal. Show the true shape of a horizontal section bisecting the axis.

QUESTIONS IN PLANE GEOMETRY.

Section 1.

(1.) Define a point. (2.) What is the true mathematical point? (3.) Define a line. (4.) What are the ends of lines called? (5.) What is the point of intersection? (6.) Name the different kinds of lines. (7.) What is a straight line? (8.) When is it said to be produced? (9.) What is a curved line? (10.) Name the directions that a curved line may have. (11.) What is a horizontal line? (12.) What is a vertical line? (13.) What is an oblique line? (14.) How many oblique lines may there be? (15.) What are parallel lines? (16.) Name the different kinds. (17.) What is an angle? (18.) On what does its magnitude depend? (19.) Name the different kinds. (20.) When is a straight line perpendicular to another? (21.) Is a perpendicular line *always* vertical? (22.) What is a right angle? (23.) Why is it made the standard for comparing other angles? (24.) What is an obtuse angle? (25.) What is an acute angle? (26.) What is a circle? (27.) Distinguish between circle and circumference. (28.) What is an arc? (29.) Define radius. (30.) What is a diameter? (31.) What is a semicircle? (32.) What is a tangent? (33.) Into how many parts is the circumference of every circle divided? (34.) What are the parts called? (35.) How many degrees in a quadrant? (36.) What relationship is there between the angles at the centre of a circle and the arcs on which they stand? (37.) What is a minute? (38.) What is a second?

Section 2.

(1.) What is Euclid's definition of a figure? (2.) What is Euclid's definition of rectilineal figures? (3.) What is a triangle? (4.) How is it the most simple of all rectilineal figures? (5.) Why is it sometimes called a trilateral? (6.) If a rectilineal figure has six sides, how many angles has it? (7.) How many kinds of triangles are there? (8.) From what are they named? (9.) What is an equilateral triangle? (10.) What is an isosceles triangle? (11.) What is a scalene triangle? (12.) What is a right-angled triangle? (13.) Which side is the hypotenuse? (14.) What are the other sides called? (15.) What is an obtuse-angled triangle? (16.) What is an acute-angled triangle? (17.) What is the vertex of a triangle? (18.) What other name has it? (19.) What is generally meant by the base? (20.) In what kind of triangles may it be changed? (21.) What is the altitude of a triangle? (22.) What is the perimeter of a figure? (23.) What is a chord?

Section 3.

(1.) What is a quadrilateral figure? (2.) What is a parallelogram? (3.) How many kinds of quadrilaterals are there? (4.) Name the four which are parallelograms. (5.) Name the two which are not parallelograms. (6.) What is a square? (7.) What is a rectangle? (8.) What other name has it? (9.) What is a rhombus? (10.) What angles are always equal to each other? (11.) What is a rhomboid? (12.) What is a trapezium? (13.) What is a trapezoid? (14.) Give another name for a quadrilateral figure. (15.) What is a diagonal? (16.) What is a diameter of *a parallelogram?*

Section 4.

(1.) What is the area of a figure? (2.) How are such measurements calculated? (3.) What is the area of a square whose side contains 6 linear inches? (4.) What is the area of a rectangle whose adjacent sides are 6 feet and 5 feet? (5.) What are concentric circles?

Section 5.

(1.) What is Euclid's definition of multilateral figures or polygons? (2.) What is a regular polygon? (3.) What is an irregular polygon? (4.) How many sides may a polygon have? (5.) What is the limit to the number of sides that we usually meet with? (6.) What is a nonagon? (7.) What name is given to a figure of eleven sides? (8.) Of twelve sides?

Section 6.

(1.) What is a cone? (2.) What is meant by its axis? (3.) When is a cone said to be right? (4.) When is it said to be oblique? (5.) Under what circumstances will a section of a cone be a circle? (6.) What is an ellipse? (7.) How many diameters has it? (8.) What is the long diameter called? (9.) What name is given to the short diameter? (10.) What are the foci? (11.) What is a parabola? (12.) What is its double ordinate? (13.) What is its ordinate? (14.) What is its abscissa? (15.) What is a hyperbola? (16.) What is its diameter? (17.) What do we mean by the "conic sections?" (18.) What is an oval? (19.) Why is it so called?

Section 7.

(1.) How many kinds of inscribed figures are there? (2.) When is a rectilineal figure said to be inscribed in another rectilineal figure? (3.) When is a rectilineal figure said to be inscribed in a circle? (4.) When is a circle said to be inscribed in a rectilineal figure? (5.) What is a sector of a circle?

Section 8.

(1.) How many kinds of described figures are there? (2.) When is a rectilineal figure said to be described about another rectilineal figure? (3.) When is a rectilineal figure said to be described about a circle? (4.) When is a circle said to be described about a rectilineal figure?

Section 9.

(1.) What is ratio? (2.) On what does the ratio of any two quantities depend? (3.) What is understood by a "part?" (4.) What is Euclid's definition of proportion? (5.) When are four quantities said to be proportionals? (6.) What is the last term called? (7.) Which are the extremes? (8.) Which are the means? (9.) What product equals the product of the means? (10.) What is meant by a mean proportional? (11.) What is a proportional in Practical Geometry? (12.) What is meant by the fourth proportional greater? (13.) What is meant by the fourth proportional less? (14.) What is meant by the third proportional greater? (15.) The third proportional less?

Section 10.

(1.) What is Euclid's definition of similar rectilineal figures? (2.) What rectilineal figures are similar? (3.) What other rectilineal figures can be made similar?

Section 11.

(1.) In Euclid I. 35, what is meant by the same parallels? (2.) In what direction is the altitude reckoned? (3.) Under what circumstances is a triangle half of a parallelogram?

QUESTIONS IN SOLID GEOMETRY.

(1.) To what has the preceding portion of the work been confined? (2.) What kind of objects next comes under consideration? (3.) What is the great difference between a plane figure and a solid object? (4.) Name the distinct ways in which a solid may be represented. (5.) Explain clearly the difference between drawing an object perspectively and geometrically. (6.) How many distinct drawings must we make in order to draw a solid object geometrically? (7.) Explain clearly what is meant by plan, and what by elevation. (8.) What do we understand by the "planes of projection?" (9.) Name them. (10.) How may the horizontal plane be illustrated? (11.) How may the vertical plane be illustrated? (12.) What do we understand by the "line of intersection?" (13.) By what other names is it known? (14.) What do we mean by the projections of an object? (15.) By what is every solid bounded? (16.) By what is every surface bounded? (17.) By what is every line limited? (18.) What do we mean by the projector of a point? (19.) How may a point be found, its projections being given? (20.) When a line is parallel to the horizontal and vertical plane, to what are its projections parallel? (21.) Under what circumstances must we suppose the vertical plane to revolve upon the line of intersection of the planes of projection? (22.) What shows the distance of a point from the horizontal plane? (23.) What shows the distance of a point from the vertical plane? (24.) What is understood by the term rabatting?

(25.) What solids are most commonly used to illustrate the principles of Solid Geometry? (26.) What is a cube? (27.) What is a prism? (28.) What is a pyramid? (29.) What is a sphere? (30.) What is a cone? (31.) What is a cylinder?

(32.) What are co-ordinate planes? (33.) What do we understand by the traces of a line? (34.) How are they distinguished? (35.) What do we understand by the traces of a plane? (36.) How are they distinguished? (37.) When the projections of a line are given, what may be found? (38.) If a plane be parallel to the ground line, to what are its traces parallel? (39.) If a trace be perpendicular to the ground line, to what are its traces perpendicular?

(40.) Name four other regular solids which are used to illustrate the principles of Solid Geometry. (41.) What is a tetrahedron? (42.) What is an octahedron? (43.) What is a dodecahedron? (44.) What is an icosahedron?

(45.) What do we understand by a section? (46.) What do we mean by penetrations of solids?

ETYMOLOGY OF GEOMETRICAL TERMS.

Abbreviations—L. for Latin; G. for Greek; F. for French.

Abscissa, from L. *abscissus, a, um*, torn off; from L. *ab-scindo*, to tear off.

Acute, from L. *acutus, a, um*, sharp or pointed; from L. *acuo*, to make sharp.

Adjacent, from L. *adjacens-entis*, lying near; from L. *ad*, to; and *jaceo*, to lie.

Alternate, from L. *alternatus, a, um;* from L. *alterno*, to do anything by turns. [L. *alter* = other.]

Altitude, from L. *altitudo, dinis*, height; from L. *altus, a, um*, high.

Angle, from L. *angulus*, a corner; from G. *angkylos*, a bend.

Apex, from L. *apex*, the *tip* or *top* of a thing.

Arc, from L. *arcus*, a bow.

Area, from L. *area*, a vacant piece of ground; originally a place where corn was *dried;* from L. *areo*, to be dry.

Axis, from L. *axis;* G. *axon*, an axle.

Base, from L. *basis*, a foundation; from G. *baino*, to step.

Bisect, from L. *bis*, twice; and L. *seco* (*sectus*), to cut.

Centre, from L. *centrum;* G. *kentron*, a sharp point. [G. *kenteo* = to prick.]

Chord, from L. *chorda;* G. *chordē*, an intestine, also a string of a lyre.

Cinquefoil, from F. *cinq*, five; and F. *feuille*, a leaf. [L. *folium* = a leaf.]

Circle, from L. *circulus*, a ring; from G. *kirkos*, a circle.

Circumference, from L. *circumfero*, to carry round; from L. *circum*, around; and L. *fero*, to carry.

Coincide, from L. *co* (*con*), together; and L. *incido*, to fall into or upon. [L. *in* = in; and L. *cado* = to fall.]

Concentric, from L. *con*, together; and L. *centrum;* G. *kentron*, a sharp point.

Cone, from L. *conus*, G. *kōnos*, that which comes to a point.

Conjugate, from L. *conjungo*, to yoke or join together; from L. *con*, together; and L. *jugum*, that which joins, a yoke.

Converge, from F. *converger*, L. *con* = with; L. *vergo*, to bend, or incline.

Co-ordinate, from L. *co* (*con*), together; and L. *ordo*, *inis*, a straight row, a regular series.

Cube, from L. *cubus;* G. *kubos*, a die.

Curve, from L. *curvus*, *a*, *um*, crooked, bent.

Cylinder, from G. *kylindros;* from G. *kylindō*, to roll. Hence *roller*-like.

Decagon, from G. *deka*, ten; and G. *gōnia*, an angle (corner).

Decimal, from L. *decem*, ten.

Diagonal, from G. *diagōnios*, from corner to corner. [G. *dia*, through; and G. *gōnia*, a corner.]

Diagram, from G. *diagramma*, that which is marked out by lines; from G. *dia*, round; and G. *graphō*, to write or delineate.

Diameter, from G. *diametros*, measurement through; from G. *dia*, through; and G. *metron*, a measure.

Dihedral, from G. *di*, double; and G. *hedra*, a base.

Dodecagon, from G. *dōdeka*, twelve; and G. *gōnia*, an angle (corner).

Dodecahedron, from G. *dōdeka*, twelve; and G. *hedra*, a base.

Duodecimal, from L. *duodecim*, twelve. [L. *duo* = two, and L. *decem* = ten.]

Elevation, from L. *elevo*, to lift up or raise; from L. *e*, up; and L. *levo*, to raise.

Ellipse, from G. *elleipsis*, a defect or leaving out. [G. *elleipo*, to leave out.]

Equilateral, from L. *aequilaterus*, equal-sided; from L. *aequus*, *a*, *um*, equal; and L. *latus*, *lateris*, a side.

Equivalent, from L. *aequus*, *a*, *um*, equal; and L. *valens*, *valentis*, being strong. [L. *valeo* = to be strong.]

Figure, from L. *figura*, a shape. [L. *fingo* = to form.]

Focus, from L. *focus*, a fire-place, hearth. [L. *foveo* = to heat.]

Foil, from F. *feuille*, a leaf. [L. *folium* = a leaf.]

Geometry, from G. *geōmetrō*, to measure land; from G. *gē*, the earth; and G. *metro*, to measure.

Heptagon, from G. *heptagōnos*, seven-cornered; from G. *hepta*, seven; and G. *gōnia*, an angle (corner).

Hexagon, from G. *hex*, six; and G. *gōnia*, an angle.

Horizontal, from G. *horizō*, to divide or bound. [G. *horos* = a limit.]

Hyperbola, from G. *hyperbolē*, a throwing beyond; from G. *hyper*, beyond or over; and *ballo*, to throw.

Hypotenuse, from G. *hypoteinousa*, the line subtending a right angle; from G. *hypo*, under; and G. *teinō*, to stretch.

Icosahedron, from G. *eikosi*, twenty; and G. *hedra*, a base.

Intersect, from L. *inter*, between; and L. *seco* (*sectum*), to cut.

Involute, from L. *involvo*, to roll around, wrap up; from L. *in*, upon; and L. *volvo*, *volutum*, to roll.

Isosceles, from G. *isoskelēs*, having equal legs; from G. *isos*, equal; and G. *skelos*, a leg.

Line, from L. *linea*, a linen thread. [L. *linum* = flax.]

Multilateral, from L. *multilaterus*, many-sided; from L. *multus*, *a*, *um*, many; and L. *latus*, *lateris*, a side.

Nonagon, from L. *nonus*, *a*, *um*, the ninth; and G. *gōnia*, a corner.

Oblique, from L. *obliquus*, *a*, *um*, sidelong, slanting.

Oblong, from L. *oblongus*, *a*, *um*, rather long.

Obtuse, from L. *obtusus*, *a*, *um*, blunt; from L. *obtundo*, to blunt; from L. *ob*, against; and L. *tundo*, to beat.

Octagon, from G. *oktō*, eight; and G. *gōnia*, an angle.

Octahedron, from G. *oktō*, eight; and G. *hedra*, a base.

Ordinate, from L. *ordo*, *inis*, a straight row, a regular series.

Oval, from L. *ovum*, an egg. Hence *egg*-shaped.

Parabola, from G. *parabolē*, a placing beside; from G. *para*, from, by the side of; and G. *ballo*, to throw.

Parallel, from G. *para*, by the side of; and G. *allelos*, one another. [G. *allos* = another.]

Parallelogram, from G. *parallēlogrammon*, a figure bounded with parallel sides. [G. *grammē* = a line.]

Penetration, from L. *penetro* (*penetratum*), to place or set into.

Pentagon, from G. *pente*, five; and G. *gōnia*, a corner.

Perimeter, from G. *peri*, around; and G. *metron*, a measure.

Periphery, from G. *peri*, around; and G. *pherō*, to carry.

Perpendicular, from L. *perpendiculum*, a plumb-line; from L. *per*, thoroughly; and L. *pendo*, to weigh.

Plane, from L. *planus*, perfectly flat.

Point, from L. *punctum*, a point, or small hole. [L. *pungo* = to pierce into]

Polygon, from G. *polugōnos*, a figure having many angles; from G. *polus*, many; and G. *gōnia*, a corner.

Prism, from G. *prisma*. [G. *prizō* = to saw.]

Problem, from G. *problēma*, that which is proposed; from G. *pro*, before; and G. *ballo*, to throw.

Profile, from L. *pro*, forth; and L. *filum*, a thread.

Projection, from L. *projicio* (*projectum*), to throw forth or before; from L. *pro*, forth; and L. *jaceo*, to throw.

Proportional, from L. *proportio*, symmetry, analogy; from L. *pro*, in comparison with; and L. *portio*, *portionis*, part, share.

Pyramid, from G. *pyramis*, *pyramidos*, usually derived from G. *pyr*, a flame, because of its pointed shape.

Quadrant, from L. *quadrans*, *-antis*, making square; from L. *quadro*, to make square. [L. *quatuor* = four.]

Quadrilateral, from L. *quadrilaterus*, four-sided; from L. *quatuor*, four; and L. *latus*, *lateris*, a side.

Quatrefoil, from F. *quatre*, four; and F. *feuille*, a leaf.

Radius, from L. *radius*, a spoke of a wheel.

Ratio, from L. *ratio*, calculation. [L. *reor* (*ratus*), to think, suppose.]

Rectangle, from L. *rectus*, *a*, *um*, right; and L. *angulus*, a corner.

Rectilineal, from L. *rectus*, *a*, *um*, right, straight; and L. *linea*, a linen thread. [L. *linum* = flax.]

Rhombus, from G. *rhombos*, a wheel thus shaped, and turned on a pivot. [G. *rhembō*, to turn round and round.]

Rhomboid, from G. *rhombos*, and G. *eidos*, form, shape.

Scale, from L. *scala*, a ladder, flight of steps. [L. *scando*, to mount.]

Scalene, from G. *skalēnos*, unequal, uneven. [G. *skazō*, to limp.]

Sector, from L. *sector*, one who cuts. [L. *seco* (*sectum*), to cut.]

Segment, from L. *segmentum*, a piece cut off. [L. *seco* (*sectum*), to cut.]

Semi-circle, from L. *semi*, half; and L. *circulus*, a ring; from G. *kirkos*, a circle.

Sphere, from G. *sphaira*, a ball or globe.

Spiral, from G. *speira*, anything wound round, a coil.

Square, from L. *quadro*, to make square. [Old F. *esquarré*, a square; modern F. *carré*, square.]

Superficies, from L. *superficies*, the upper side, the top; from L. *super*, above; and L. *facies*, a face.

Tangent, from L. *tangens*, *-entis*, touching. [L. *tango* = to touch.]

Tetrahedron, from G. *tetra*, four; and G. *hedra*, a base.

Trace, from L. *traho*, to draw. [F. *trace* = trace.]

Transverse, from L. *transversus*, lying across; from L. *trans*, across; and L. *verto* (*versum*), to turn.

Trapezium, from G. *trapezion*, dim. of *trapeza*, a table; contracted either from G. *tri-peza*, three-legged; or from G. *tetra-peza*, four-legged.

Trapezoid, from G. *trapezion*, and G. *eidos*, shape, form.

Trefoil, from F. *trois*, three; and F. *feuille*, a leaf.

Triangle, from L. *triangulus*, having three corners; from L. *tres*, three; and L. *angulus*, a corner.

Trilateral, from L. *trilaterus*, having three sides; from L. *tres*, three; and L. *latus*, *lateris*, a side.

Trisect, from L. *tres*, three; and L. *seco* (*sectum*), to cut.

Vertex, from L. *vertex*, the top or crown of the head. [L. *verto* = to turn.]

Vertical, from L. *vertex* (*verticis*), the top of the head.

Volute, from L. *volvo* (*volutum*), to roll.

INDEX TO PROBLEMS IN PLANE GEOMETRY.

PAGE

Altitude; to find, of given triangle, ... 25

Angle; to bisect, ... 10

—— of 45°; to construct, on given line, ... 11

—— of 60°, 30°, or 15°; to construct, on given line, ... 11

——; to make, equal to given angle, from given point, *in* the given line, ... 13

——; to make, equal to given angle, from given point, *outside* the given line, ... 14

——; to bisect, made by two converging lines, the angular point being inaccessible, ... 14

——, **right**, to trisect, ... 10

Arc; to bisect, ... 6

——, **tangential**; to draw, to two given circles, touching one of the given circles in a given point, ... 54

—— **of a circle**; to describe, tangential to two given converging lines, ... 55

——; to describe, tangential to given arc and given line, ... 170

Circle; to find centre of, ... 45

——; to describe, passing through three given points, ... 45

——; to describe, passing through given point, and tangential to given line, ... 46

——; to describe, touching two given circles, ... 48

——; to describe, of given radius, touching two given circles, ... 49

——; to inscribe, within given triangle, ... 98

——; to inscribe, within given square, ... 98

——; to inscribe, within given rhombus, ... 99

——; to inscribe, within given trapezium, ... 99

——; to inscribe, within given quadrant, ... 100

——; to describe, about given triangle, ... 120

——; to describe, about given square, ... 121

——; to construct, two-thirds of given circle, ... 164

——; to describe, of given radius, tangential to two converging lines, 169

——; to draw, touching given circle in given point, also a given line, 171

PAGE

Circle; to inscribe, in given angle, passing through given point, ... 172
——; to construct, touching given line in given point, also given smaller circle, ... 173
——; to construct, passing through two given points, and touching given line, ... 174
——; to draw, tangential to two unequal circles, ... 175
——; to divide, into any number of parts, equal in area and outline, 47
——, **area of**; to divide, into any number of parts by concentric circles, ... 46
Circles, three; to describe, having given radii, each being tangential to the other two, ... 49
——; to inscribe, in given triangle, each touching two others, and two sides of triangle, ... 105
——; to inscribe, in given triangle, each touching other two, and one side of triangle, ... 106
——; to inscribe, within given equilateral triangle, touching each other, and two sides of triangle, ... 107
——; to inscribe, within given circle, ... 107
——; to inscribe, within given equilateral triangle, each touching other two, and one side of triangle, ... 168
Circles, four; to inscribe, in given square, touching each other, and one side only of square, ... 108
——; to inscribe, in given square, touching each other, and two sides of given square, ... 109
——; to inscribe, in given octagon, ... 110
——; to inscribe, in given circle, ... 111
Circles, five; to inscribe, in given circle, ... 112
Circles, six; to inscribe, within equilateral triangle, ... 113
——; to describe, about and equal to given circle, ... 121
Circles, seven; to inscribe, in given circle, ... 113
Circles, any number of; to inscribe, in given circle, ... 103
——; to describe, about a given circle, ... 122
Circles, series of; to describe, tangential to two converging lines, ... 50
Decagon; to inscribe, within given circle, ... 62
——; to construct, on given line, ... 69
Dodecagon; to inscribe, within given circle, ... 64
Ellipse; to describe, transverse and conjugate diameters given, ... 75-79
——; to find centre and axes of, ... 79
——; to complete curve of, one quarter given, ... 82
——; to complete curve of, more than half given, ... 83
——; to describe, about rectangle, ... 83
Elliptical figure; to describe, one diameter given, ... 80
——; to construct, two squares given, ... 80
Equilateral triangle; to construct, on given base, ... 23
——; to construct, height given, ... 24
——; to inscribe, within given circle, ... 88
——; to inscribe, in given square, ... 89

PAGE

Equilateral triangle; to inscribe, in given hexagon, ... 89
——; to inscribe, in given pentagon, ... 90
——; to describe, about given square, ... 116
——; to describe, about given circle, ... 117
——; to construct, equal to given triangle not equilateral, ... 155
Foiled figure; to construct, about any polygon, ... 123
Four-sided equilateral figure; to inscribe, in a given parallelogram, 96
Heptagon; to inscribe, within given circle, ... 60
——; to construct, on given line, ... 67
Hexagon; to inscribe, within given circle, ... 59
——; to construct, on given line, ... 66
——; to construct, diameter given, ... 71
Hyperbola; to construct, diameter, abscissa, and ordinate given, ... 86
Involute of circle, to construct, ... 179
Ionic volute, to construct, longest diameter given, ... 180
Isosceles triangle; to construct, given base and vertical angle, ... 29
——; to construct, given base and altitude, ... 30
——; to construct, given base and vertical angle of 90°, ... 30
——; to construct, given base and vertical angle containing a required number of degrees, ... 31
——; to construct, given one of equal sides and one of equal angles, 31
——; to construct, given base and perimeter, ... 32
——; to inscribe, within given square, having given base, ... 90
——; to construct, equal in area to, and on one side of, given square, ... 154
——; to construct, equal in area to given triangle, ... 155
——; to construct, given altitude and length of its equal sides, ... 167
Line; to draw, parallel to given line, at given distance from it, ... 12
——; to draw, parallel to given line, through given point, ... 13
——; to divide, into any number of equal parts, ... 16, 17
——; to divide, proportionally to given divided line, ... 17, 18
——; to draw, through given point, which, if produced, would pass through the angular point, towards which the given lines converge, ... 167
Lines; to draw, from any two given points, outside a given line, to make equal angles with the given line, ... 15
——; to draw, from any two given points, outside a given line, so that they may be equal in length, ... 15
——; to draw, from any two given points, going to the same point to which lines converge, where the point of convergence is inaccessible, ... 18
——, **parallel**; to join the extremities of, by a pair of arcs, ... 173
Nonagon; to inscribe, within given circle, ... 61
——; to construct, on given line, ... 68
Octagon; to inscribe, within given circle, ... 60
——; to construct, on given line, ... 67
——; to inscribe, in given square, ... 97

PAGE

Octagon; to construct, within given circle, making one of its angles coincide with a given point, ... 169
Oval; to describe, by arcs of circles, ... 84
Parabola; to construct, its ordinate and abscissa being given, ... 85
Parallelogram; to construct, on given base equal in area to given parallelogram, ... 140
——; to construct, equal to given triangle in area and perimeter, ... 144
——; to divide, into two parts, proportionate in area to given divided line, ... 145
Pentagon; to inscribe, within given circle, ... 58
——; to construct, on given line, ... 65
——; to describe, about a given pentagon, sides parallel, and equal to given line, ... 118
Perpendicular; to draw, to a given straight line, from given point in the line, ... 6, 7
——; to draw, to a given straight line, from given point outside it, ... 8, 9
——; to draw, to curve of ellipse, from given point, ... 81
Point; to find, which would be situated in the continuation of arc, centre of arc being not obtainable, ... 54
Polygon, any; to inscribe, in given circle, ... 57, 58
——; to construct, on given line, ... 64, 65
——; to complete, two sides being given, ... 71
——; to construct, about given circle, ... 120
——; to construct, equal in area to given triangle, ... 163
Polygon, irregular; to make, equal to given irregular polygon, ... 146
Proportional; to find a mean, between two given lines, ... 125
——, **third**; to find, to two given lines (less), ... 127
——; to find, to two given lines (greater), ... 128
——, **fourth**; to find, to three given lines (less), ... 126
——; to find, to three given lines (greater), ... 126
Rectangle; to construct, two sides given, ... 38
——; to construct, one side and diagonal given, ... 38
——; to inscribe, in given triangle, side equal to given line, ... 96
——; to construct, similar to given rectangle, ... 135
——; to construct, equal in area to given square, ... 140
——; to construct, equal in area to given triangle, ... 141
——; to construct, equal in area to given rectangle, ... 141
——; to construct, having a given side, equal in area to a given triangle, ... 142
——; to construct, equal in area to given circle, ... 144
Rectilineal figure; to convert, into triangle of equal area, ... 160.
——; to reduce, to an equivalent figure, having a less number of sides, ... 161
——; to construct, area being given in proportion to another rectilineal figure of the same kind, ... 165
——; to change, into another rectilineal figure of equal area, but having one side more, &c., ... 175

PAGE

Reduced copy; to make, of any given figure, ... 176
Rhomboid; to construct, two adjacent sides and diagonal given, ... 40
——; to construct, two adjacent sides and angle given, ... 41
Rhombus; to construct, side and diagonal given, ... 39
——; to construct, side and angle given, ... 40
Semicircles, three; to inscribe, within equilateral triangle, having adjacent diameters equal (**The Trefoil**), ... 100
——; to inscribe, within given circle, ... 101
Semicircles, four; to inscribe within given square, each touching two sides of square (**The Quatrefoil**), ... 102
——; to inscribe within given square, each touching one side of the square, ... 103
Semicircles, any number of; to inscribe, in given circle, ... 104
Spiral; to construct of one revolution, ... 178
——, **common**; to construct, on given diameter, ... 177
Square; to construct, on given base, ... 36
——; to construct, diagonal given, ... 37
——; to inscribe, within circle, ... 91
——; to inscribe, within triangle, ... 92
——; to inscribe, within rhombus, ... 92
——; to inscribe, within trapezium, ... 93
——; to inscribe, within pentagon, ... 94
——; to inscribe, within hexagon, ... 94
——; to inscribe, within quadrant, ... 95
——; to describe, about circle, ... 119
——; to construct, within given square, concentric with it, ... 134
——; to construct, equal in area to given rectangle, ... 147
——; to construct, having any number of square inches, ... 147
——; to construct, equal in area to given triangle, ... 148
——; to construct, area one-third greater than given square, ... 148
——; to construct, area one-third less than given square, ... 149
——; to inscribe, within given square, proportional in area, ... 150
——; to construct, equal in area to trapezium, ... 151
——; to construct, equal in area to any number of squares, ... 152
——; to construct, equal in area to any given polygon, ... 153
Straight line; to bisect, ... 6
——; to divide, into extreme and mean proportion, ... 128
——; to divide, successively into half, third, &c., ... 129
——; to divide, in point, ratio 2 : 3, ... 129
——; to divide, in point; whole : part : : 5 : 3, ... 130
——; to draw, equal to half circumference of circle, ... 143
Tangent; to draw, to given circle, point of contact in circumference, 51
——; to draw, to given circle, point outside circumference, ... 51
——; to draw, on outside of two given circles, placed apart, ... 52
——; to draw, between two given circles, placed apart, ... 53
——; to draw, to point of contact in arc of circle, where the centre cannot be obtained, ... 53

PAGE

Tangent; to draw, to curve of ellipse, at given point of contact, ... 82
Tangents, exterior; to draw one or two, common to two given circles, ... 171
Trapezium; to construct, equal to given trapezium, ... 41
——; to construct, adjacent pairs of sides given, and diagonal, ... 42
——; to construct, diagonal and angles at extremities given, ... 43
——; to construct, similar to given trapezium, ... 136
Triangle; to construct, three sides given, ... 24
——; to find centre of, ... 26
——; to construct, base and angles at base given, ... 26
——; to construct, altitude and angles at base given, ... 27
——; to construct, base, altitude, and one side given, ... 27
——; to construct, given base, angles of 60°, 30°, and 90°, ... 28
——; to bisect, by line parallel to one of its sides, ... 29
——; to describe, about given circle, angles being equal to those of given triangle, ... 117
——; to inscribe, within, and equidistant, and similar to, given triangle, ... 131
——; to describe, about, and equidistant, and similar to, given triangle, ... 132
——; to inscribe, in circle, similar to given triangle, ... 133
——; to construct, similar to given triangle, perimeter equal to given straight line, ... 133
——; to inscribe, within given triangle, similar to another triangle, 134
——; to construct, equal in area to given trapezium, ... 151
——; to construct, on given base, area equal to given triangle, ... 156
——; to construct, similar to given triangle, twice its area, ... 156
——; to bisect, line perpendicular to one of its sides, ... 157
——; to construct, equal in area to two dissimilar triangles, ... 158
——; to divide, into any number of equal parts, by lines drawn from each angle to a point within the triangle, ... 158
——; to divide, into any number of equal parts, by lines drawn from a given point in one of its sides, ... 159
——; to divide, into any number of equal parts, by lines drawn from a given point within the triangle, ... 160
——; to construct, equal in area to any given polygon, ... 162
——; to construct, equal in area to any given circle, ... 163
Undecagon; to inscribe, within given circle, ... 63
——; to construct, on given line, ... 70

INDEX TO PROBLEMS IN SOLID GEOMETRY.

PAGE

Angle; to determine between two planes, ... 219
Cone; to find its plan, elevation being given, ... 208
——; to find its elevation, plan being given, ... 209
——; to project, whose axis is inclined at angle of 45° to the horizontal plane, but parallel to the vertical, ... 237
——; to draw the section of, when cut by a plane parallel to its base, ... 245
——; to draw the sectional elevation of, when cut by a vertical plane, ... 255
——; to find the projection of, standing on its base, when the section plane is perpendicular to the vertical plane, and making an angle with the horizontal plane; also a projection of the cone, showing the true form of the section, ... 256
——; to find the projection of, standing on its base, when the section plane is parallel to one side of the cone, and perpendicular to the vertical plane, also the true form of the section, ... 258
Cube; to find its plan, elevation being given, ... 200
——; to find its plan, elevation being given, one face inclined to the ground at an angle of 60°, another face at an angle of 30°, ... 200
——; to find its elevation, plan being given, ... 201
——; to construct the projections of, having a face and one of its edges inclined at given angles, ... 227
——; to find the projection of, when one of its diagonals is perpendicular to the plane of projection, ... 228
——; to find section of, ... 247
Cylinder; to find the plan of, elevation being given, ... 207
——; to find the elevation of, plan being given, ... 208
—— **section**; to draw, when lying on the ground with its ends parallel to the vertical plane, and at right angles to the horizontal plane; the plane of section being parallel to the axis of the cylinder, ... 247
——; to draw the curve of penetration of two, whose axes are at right angles to each other, ... 260

PAGE

Cylinder section; to draw the curve of penetration of two, whose diameters are equal, and whose axes are at right angles to each other, ... 261

——; to draw the curve of penetration of two, whose diameters are equal, and whose axes are at right angles to each other, one of the cylinders being inclined to the vertical plane, ... 262

Dodecahedron; to draw the plan and elevation of, when one edge of its base is inclined at an angle of 30° to the ground line, ... 243

Equilateral triangle; to draw the plan of, two of its sides inclined at 60° and 30° to the horizon, and to determine the inclination of the plane in which it is situated, ... 222

Line; to draw its plan, elevation given at right angles to the vertical plane, ... 194

——; to find the plan of, its elevation being given parallel to the two planes of projection, ... 195

——; to find the plan of, its elevation being given parallel to the horizontal plane, but inclined to the vertical plane of projection, ... 196

——; to find the plan of, inclined to both planes of projection, its elevation being given, ... 196

——; to find elevation of, its plan being given, ... 198

——; traces of, given, to find its projections, ... 213

——; given, projections, to find its length, ... 213

——; given, projections of, to find the angles which it makes with the planes of projection, ... 214

——, **any**; to determine the plan and elevation of, inclined at 60° to the horizontal plane, and 20° to the vertical plane, ... 221

Octahedron; to construct the projections of, when its axis is vertical, ... 239

——; to construct the projections of, when it lies on its face on the horizontal plane, ... 241

——; to construct the plan and elevation of, when resting on one of its faces, and when one edge of this face makes an angle of 15° with the vertical plane, ... 242

Parallel planes; traces of being given, to find the distance between them, ... 217

——; to determine, by traces, and given distance, ... 218

Pipe, hollow; to draw section of, ... 246

Plane; to draw, so that it makes a given angle with a given plane, and passes through a line in the first, ... 220

——; to determine by its traces, containing three given points, ... 224

Planes, two; traces of, being given, to find the projections of their common intersection, ... 215

Point; to find the plan of, elevation being given, ... 194

Prism, triangular; to project the section of, when cut by an oblique plane, ... 248

PAGE

Prism, square; to draw the plan of, its elevation being given, ... 202
——, **pentagonal**; to draw the plan of, its elevation being given, ... 203
——, **hexagonal**; to find the elevation of, its plan being given, ... 204
——; to draw the plan and elevation of, which has its axis inclined 40° to the paper, and one face parallel to the vertical plane, 230
——; to find the plan and elevation of, the height of which is 30′, and length of one side 10′, on a scale of 20′ to the inch; the front face being parallel to the vertical plane, ... 232
Pyramid, square; to find the elevation of, its plan being given, ... 206
——; to find the vertical projection of the section, parallel to the section plane, the trace of the plane, which is vertical, being at right angles to the plan of one of the lateral edges of the pyramid, ... 252
——; to draw the horizontal projection of the section of, when cut by a plane parallel to its base, the plane of the base of the pyramid being inclined at 40°, ... 254
——, **pentagonal**; to draw the plan of, when one edge of its base is inclined at an angle of 45°, ... 233
——; to construct the sectional plan and elevation of, when standing on its base on the horizontal plane, ... 250
——; to construct the sectional elevation and plan of, when standing on its base on the horizontal plane, ... 251
——, **hexagonal**; to find the plan of, its elevation being given, ... 204
——; to determine the plan of, when lying on one of its faces on the horizontal plane, ... 234
——; to project, when its axis is inclined at an angle of 50° to the horizontal plane, but parallel to the vertical, ... 235
Rectangular surface; to find the plan of the end elevation of, when given parallel to the horizontal plane, ... 197
Solid, any; the projections of, being given, to determine other projections from them, ... 236
Sphere; to find both plan and elevation of, ... 210
Square; to draw the plan of, when its surface is inclined 42°, and one of its sides is horizontal, ... 229
Straight lines, two; to determine the angle contained by, given by their projections, ... 216
Tetrahedron; to construct the projections of, ... 239
——; to construct the sectional elevation of, upon a vertical plane parallel to the section plane, the trace of which is at right angles to one of the lateral edges of the solid, ... 249
Traces of plane; being given, to find the angles which it makes with the planes of projection, ... 223

PRINTED BY G. PHILIP AND SON, LIVERPOOL.